KEY pg 1

1. a. False due to b_3

b. False $b_5 = 3b_1 + b_2 + b_4$

c. False (see b.)

d. True. any vector can be written as a linear combination of b_1, b_2, b_4, b_5, b_6:

$$\begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 3 \\ 2 \\ 23 \\ 9 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$$

Using $b_5 = 3b_1 + b_2 + b_4$

$$= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix} + \alpha_4 \left(3 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix} \right) + \alpha_5 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$$

$$= (\alpha_1 + 4\alpha_4) \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + (\alpha_2 + \alpha_4) \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + (\alpha_3 + \alpha_4) \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$$

$$= \beta_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix} + \beta_4 \begin{bmatrix} 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$$

So the only question is if $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} \right\}$ is a basis. It is.

e. False. not linearly independent

f. True. (see d.)

g. True. Check the determinant.

Key - pg 2

1h. True. Since $\{b_5, b_6, b_7, b_8\}$ does, this set does too.

i. False. Linearly dependent set

j. False

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 7 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ 4 \\ 4 \\ -4 \end{bmatrix}$$

or

$$\left.\begin{aligned} \alpha_1 + 4\alpha_3 &= 1 \\ 4\alpha_3 &= 1 \\ 3\alpha_1 + 4\alpha_3 &= 1 \\ -\alpha_1 - 4\alpha_3 &= 7 \end{aligned}\right\} \text{inconsistent}$$

K. False.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 2 \\ 23 \\ 9 \end{bmatrix}$$

l. True Can't write any 1 as sum of others

m. False

$$\begin{bmatrix} 4 \\ 4 \\ 4 \\ -4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ -1 \\ 7 \\ 4 \end{bmatrix} + \gamma \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\left.\begin{aligned} \alpha - \beta + 4\gamma &= 4 \\ -\beta + 4\gamma &= 4 \end{aligned}\right\} \Rightarrow \alpha = 0$$

$$3\alpha + 7\beta + 4\gamma = 4$$

$$-\alpha + 4\beta + 4\gamma = -4$$

$$-\beta + 4\gamma = 4$$

$$7\beta + 4\gamma = 4$$

$$\Rightarrow 6\beta = 0 \text{ or } \beta = 0$$

$$\Rightarrow \gamma = 1$$

$$-\alpha + 4\beta + 4\gamma = 4 \text{ not } -4$$

Inconsistent equations

Key - pg 3

1 a. True. Check the determinant

2. $V_1 = b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$V_2 = b_2 - \frac{\langle V_1, b_2 \rangle}{\langle V_1, V_1 \rangle} V_1$$

$$\langle V_1, b_2 \rangle = 1 \qquad \langle V_1, V_1 \rangle = 4$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -1/4 \\ -1/4 \\ 3/4 \end{bmatrix}$$

$$V_3 = b_3 - \frac{\langle V_1, b_3 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle V_2, b_3 \rangle}{\langle V_2, V_2 \rangle} V_2$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{(-1)}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5/4}{12/16}\begin{bmatrix} -1/4 \\ -1/4 \\ -1/4 \\ 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} - \begin{bmatrix} -5/12 \\ -5/12 \\ -5/12 \\ 15/12 \end{bmatrix} = \begin{bmatrix} -24/24 \\ -24/24 \\ 0/24 \\ 24/24 \end{bmatrix} + \begin{bmatrix} 6/24 \\ 6/24 \\ 6/24 \\ 6/24 \end{bmatrix} - \begin{bmatrix} -10/24 \\ -10/24 \\ -10/24 \\ 30/24 \end{bmatrix}$$

$$= \begin{bmatrix} -8/24 \\ -8/24 \\ 16/24 \\ 0 \end{bmatrix}$$

Key - pg 4

$$v_4 = b_4 - \frac{\langle v_1, b_4\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle v_2, b_4\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle v_3, b_4\rangle}{\langle v_3, v_3\rangle} v_3$$

$$= \begin{bmatrix}1\\3\\1\\4\end{bmatrix} - \frac{9}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{7/4}{12/16}\begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix} - \frac{-16/24}{(384/576)}\begin{bmatrix}-8/24\\-8/24\\16/24\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\3\\1\\4\end{bmatrix} - \frac{9}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{7}{3}\begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix} + \begin{bmatrix}-8/24\\-8/24\\+16/24\\0\end{bmatrix}$$

$$= \begin{bmatrix}12/12\\36/12\\12/12\\48/12\end{bmatrix} - \begin{bmatrix}27/12\\27/12\\27/12\\27/12\end{bmatrix} - \begin{bmatrix}-7/12\\-7/12\\-7/12\\21/12\end{bmatrix} + \begin{bmatrix}-4/12\\-4/12\\+8/12\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

Now form the orthonormal set

$$\hat{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \hat{v}_2 = \frac{1}{\sqrt{12/16}}\begin{bmatrix}-1/4\\-1/4\\-1/4\\3/4\end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix}-1\\-1\\-1\\3\end{bmatrix}$$

$$\hat{v}_3 = \frac{1}{\sqrt{384}}\begin{bmatrix}-8/24\\-8/24\\16/24\\0\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\\0\end{bmatrix}$$

$$\hat{v}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

Key - pg 5

3a. $y_1 = Ax_1 = \begin{bmatrix} 12 & 0 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \end{bmatrix}$

$y_2 = Ax_2 = \begin{bmatrix} 12 & 0 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix} = \begin{bmatrix} -84 \\ 49 \end{bmatrix}$

b.
$$\begin{aligned} A_{New} &= C^{-1} A_{old}\, C \\ &= \begin{bmatrix} 1 & 7 \\ -3 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 12 & 0 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ -3 & -4 \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} -4 & -7 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ -3 & -4 \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} -27 & -49 \\ 33 & 7 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ -3 & -4 \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} 120 & 7 \\ 12 & 203 \end{bmatrix} \end{aligned}$$

c. $x_{1new} = C^{-1} x_1 = \frac{1}{17} \begin{bmatrix} -4 & -7 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} -25 \\ 6 \end{bmatrix}$

$x_{2new} = C^{-1} x_2 = \frac{1}{17} \begin{bmatrix} -4 & -7 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 0 \\ -17 \end{bmatrix}$

Key - pg 6

3d. $y_{1new} = A_{new}\, x_{1new} = \frac{1}{17}\begin{bmatrix} 120 & 7 \\ 12 & 203 \end{bmatrix} \cdot \frac{1}{17}\begin{bmatrix} -25 \\ 6 \end{bmatrix}$

$$= \frac{1}{289}\begin{bmatrix} -2958 \\ 918 \end{bmatrix} = \begin{bmatrix} -10.2353 \\ 3.1765 \end{bmatrix}$$

$y_{2new} = A_{new}\, x_{2new} = \frac{1}{289}\begin{bmatrix} 120 & 7 \\ 12 & 203 \end{bmatrix}\begin{bmatrix} 0 \\ -17 \end{bmatrix}$

$$= \frac{1}{289}\begin{bmatrix} 119 \\ 3451 \end{bmatrix} = \begin{bmatrix} -0.4118 \\ -11.9412 \end{bmatrix}$$

3e. $y_{1new} = C^{-1} y_1 = \frac{1}{17}\begin{bmatrix} -4 & -7 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 18 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -174 \\ 54 \end{bmatrix}$

$$= \begin{bmatrix} -10.2353 \\ 3.1765 \end{bmatrix} \checkmark$$

$y_{2new} = C^{-1} y_2 = \frac{1}{17}\begin{bmatrix} -4 & -7 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -84 \\ 49 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -7 \\ -203 \end{bmatrix}$

$$= \begin{bmatrix} -0.4118 \\ -11.9412 \end{bmatrix}$$

Key - pg 7

4 a. Dimension $= 2$

b. $\begin{bmatrix} 1 \\ 3 \\ 15 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \alpha - \beta \\ \beta \\ 3\alpha + \beta \end{bmatrix} \Rightarrow \beta = 3$ and $\alpha = 4$

yes.

c. $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \alpha - \beta \\ \beta \\ 3\alpha + \beta \end{bmatrix} \Rightarrow \beta = 1, \alpha = 1$

$\rightarrow$ inconsistent

no.

d. all vectors of the form $\begin{bmatrix} \alpha - \beta \\ \beta \\ 3\alpha + \beta \end{bmatrix}$ are in the span.

Just use the equation from b!

Key - pg 8

5. $\begin{bmatrix} 1 & 3 & 4 \\ 1 & -7 & -6 \\ 5 & 4 & 9 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

① $u + 3v + 4w = 0$
② $u - 7v - 6w = 0$
③ $5u + 4v + 9w = 0$

① - ② $\quad -10v - 10w = 0 \quad$ or $\quad v = -w$

③ $\quad 5u + 4(-w) + 9w = 0$
$\quad 5u + 5w = 0 \quad$ or $\quad u = -w$

$\therefore \quad u = v$

all vectors of the form $\begin{bmatrix} u \\ u \\ -u \end{bmatrix}$

are in the null space. $\therefore$ $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is

and $\begin{bmatrix} 7 \\ 7 \\ -7 \end{bmatrix}$ is, but $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not.

Key - pg 9

6 a) $\lambda I - A = \begin{bmatrix} \lambda-1 & 0 \\ -1 & \lambda-3 \end{bmatrix}$

$|\lambda I - A| = (\lambda-1)(\lambda-3)$

so eigenvalues are $\lambda = 1$, $\lambda = 3$

eigenvector corresponding to $\lambda = 1$ solves

$\lambda e = A e$, $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$e_1 = e_1$
$e_2 = e_1 + 3e_2$

Choose $e_1 = 1$ then $e_2 = -1/2$
$= \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

eigenvector corresponding to $\lambda = 3$
$\begin{bmatrix} 3e_1 \\ 3e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$3e_1 = e_1$ so $e_1 = 0$
$3e_2 = e_1 + 3e_2$ Choose $e_2 = 1$

Key - pg 10

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the eigenvectors are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

b. $(A-1)(A-3) = A^2 - 4A + 3I$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 9 \end{bmatrix}$$

$$-4A = \begin{bmatrix} -4 & 0 \\ -4 & -12 \end{bmatrix} \qquad 3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ -4 & -12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c. $C = \begin{bmatrix} 0 & 1 \\ 1 & -1/2 \end{bmatrix}, \quad C^{-1} = \frac{1}{-1}\begin{bmatrix} -1/2 & -1 \\ -1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in basis C is $C^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} +7/2 \\ +1 \end{bmatrix}$

Key - pg 11

$$C^{-1}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2.5\\-1\end{bmatrix}$$

$$C^{-1}\begin{bmatrix}11\\7\end{bmatrix} = \begin{bmatrix}25/2\\-11\end{bmatrix}$$