1. $\langle V, W \rangle = V^T W = 0$

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ 3 \\ 1 \end{bmatrix} = \alpha + 6 + 1$$

$$= \alpha + 7 = 0 \quad \text{so} \quad \alpha = -7.$$

2. Can I write

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} ?$$

$$\left.\begin{aligned} 1 &= 1\alpha_1 + 0\,\alpha_2 \\ 2 &= 0\alpha_1 + 1\alpha_2 \\ 3 &= 0\,\alpha_1 + 1\,\alpha_2 \end{aligned}\right\} \text{ inconsistent}$$

also note

$$\left| \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \right| = 1$$

So vectors are linearly independent.

3. Find if they are linearly dependent

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(1) $1 = 0\,\alpha_1 - \alpha_2 + \alpha_3$
(2) $1 = 0\,\alpha_1 + \alpha_2 + 0\alpha_3$
(3) $1 = 0\,\alpha_1 + 3\,\alpha_2 + 1\alpha_3$
(4) $1 = 1\,\alpha_1 + 1\alpha_2 + 0\,\alpha_3$

from (2), $\alpha_2 = 1$

from (4), $\alpha_1 = 0$

from (3), $\alpha_3 = -2$

from (1), then $1 = 0 \cdot 0 - 1 - 2$
$1 \neq -3$

inconsistant eqns., linearly independent.

note $\left| \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \right| = 4$

3b. dim = 4

3c. yes, 4 linearly independent 4-d vectors must span the space.

in fact $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 1.25x_1 - x_2 - .25x_3 - .5x_4$

4. $A\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 11 \\ 4 \end{pmatrix} = ?$ note $\begin{pmatrix} 11 \\ 4 \end{pmatrix} = 1.76\begin{pmatrix} 5 \\ 6 \end{pmatrix} - 2.19\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

so $A\begin{pmatrix} 11 \\ 4 \end{pmatrix} = 1.76\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2.19\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -7 \\ -0.43 \end{pmatrix}$

5. Can I write $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$?

$$\begin{cases} \alpha_1 = 1 \\ -2\alpha_1 + \alpha_2 = 2 \Rightarrow \alpha_2 = 4 \\ 3\alpha_1 + \alpha_2 = 3 \Rightarrow \alpha_2 = 0 \end{cases} \Bigg] \text{inconsist}$$

5. cont.

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \frac{3}{7\!\!\!/1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/7 \\ -20/7 \\ 12/7 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4/7 \\ -20/7 \\ 12/7 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 4/7 \\ -20/7 \\ 12/7 \end{bmatrix}, \begin{bmatrix} 4/7 \\ -20/7 \\ 12/7 \end{bmatrix} \right\rangle} \begin{bmatrix} 4/7 \\ -20/7 \\ 12/7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{-8/7}{\frac{560}{49}} \cdot \frac{1}{7} \begin{bmatrix} 4 \\ -20 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{70} \begin{bmatrix} 4 \\ -20 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{50}{140} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{2}{140} \begin{bmatrix} 4 \\ -20 \\ 12 \end{bmatrix}$$

$$= \frac{1}{140} \begin{bmatrix} -42 \\ 0 \\ 14 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Step 2 - create unit vectors

$$\hat{V}_1 = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\hat{V}_2 = \frac{1}{\sqrt{560}} \begin{bmatrix} 4 \\ -20 \\ 12 \end{bmatrix} \qquad \hat{V}_3 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

6. $Z = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$

$$\alpha_1 = \langle Z, \hat{V}_1 \rangle = 1.3363$$
$$\alpha_2 = \langle Z, \hat{V}_2 \rangle = -3.8877$$
$$\alpha_3 = \langle Z, \hat{V}_3 \rangle = -6.6408$$

$$Z = 1.3363\,\hat{V}_1 - 3.8877\,\hat{V}_2 - 6.6408\,\hat{V}_3$$

$$> \begin{bmatrix} 1.3363 \\ -3.8877 \\ -6.6408 \end{bmatrix} \left[\hat{V}_1 \;\; \hat{V}_2 \;\; \hat{V}_3 \right]$$

7. $A\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$ $\quad A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ $\quad A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\5\end{pmatrix}$

Find $A\begin{pmatrix}u\\v\\w\end{pmatrix}$. First, write $\begin{pmatrix}u\\v\\w\end{pmatrix}=\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$\begin{cases} u=\alpha_1+\alpha_2+\alpha_3 \\ v=\alpha_2+\alpha_3 \\ w=\alpha_1+\alpha_3 \end{cases}$$

or $\alpha_2 = v-\alpha_3$

$u=\alpha_1+v \qquad \underline{\alpha_1 = u-v}$

$w=\alpha_1+\alpha_3 = (u-v)+\alpha_3 \qquad \underline{\alpha_3 = w-u+v}$

$\alpha_2 = v-\alpha_3 = v-(2v-u) = \underline{-u-w}$

$\therefore$

$$A\begin{pmatrix}u\\v\\w\end{pmatrix} = A\left[\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha_3\begin{pmatrix}1\\1\\1\end{pmatrix}\right]$$

$$=(u-v)\begin{pmatrix}2\\-1\\3\end{pmatrix}+(u-w)\begin{pmatrix}-1\\-1\\2\end{pmatrix}+(w-u+v)\begin{pmatrix}1\\1\\5\end{pmatrix}$$

$$= \begin{pmatrix} 2u - 2v + u + w + w - u + v \\ -u + v - u + w + w - u + v \\ 3u - 3v + 2u - 2w + 5w - 5u + 5v \end{pmatrix}$$

$$= \begin{pmatrix} 0u + 2w - v \\ -3u + 2w + 2v \\ 0u + 3w + 2v \end{pmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ -3 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$A \uparrow$