1\. a. $\int_0^1 c(2-x)\,dx = \left(2cx - \frac{cx^2}{2}\right)\Big|_0^1 = 2c - \frac{1}{2}c = 1$

$\underline{c = 2/3}$

b. $F(x) = \int_0^x \frac{2}{3}(2-x)\,dx = \frac{4}{3}\cdot x - \frac{1}{3}x^2$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{4}{3}x - \frac{1}{3}x^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

c. $F(2.6) = 1$. There is a 100% probability $x \le 2.6$.

d. $F(0) = \frac{4}{3}(0) - \frac{1}{3}(0)^2 = 0$

$F(\infty) = \int_0^1 \frac{2}{3}(2-x)\,dx = 1$

e. $P\{1 \le x \le 5\} = F(5) - F(1) = 0$

2\. $F(x) = \begin{cases} c+x & -1 < x < 0 \\ c-x & 0 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$

a. $\int_{-1}^0 (c+x)\,dx + \int_0^1 c(-x)\,dx$

$= \left(cx + \frac{x^2}{2}\right)\Big|_{-1}^0 + \left(cx - \frac{x^2}{2}\right)\Big|_0^1$

$= (0) - (-c + 1/2) + (c - 1/2)$

$= (c - 1/2) + (c - 1/2) = 2c - 1 \quad \underline{c = 1}$

b. $F(x) = \int_{-1}^x (1+x)\,dx$ on $-1 < x < 0$

$= x + \frac{x^2}{2}\Big|_{-1}^x = \left(x + \frac{x^2}{2}\right) - \left(-1 + 1/2\right)$

$= \frac{x^2}{2} + x + \frac{1}{2}$

$$F(x) = \int_{-1}^{0} (1+x)\,dx + \int_{0}^{x} (1-x)\,dx \quad \text{on } 0 \le x < 1$$

$$= \left(x + \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(x - \frac{x^2}{2}\right)\Big|_{0}^{x}$$

$$= (0) - (-1 + 1/2) + \left(x - \frac{x^2}{2}\right)$$

$$= \frac{1}{2} + x - \frac{x^2}{2}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2} & -1 \le x < 0 \\ -\frac{x^2}{2} + x + \frac{1}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

c. $F(2.6) = 1$

d. $F(-\infty) = 0$, $F(+\infty) = 1$

e. $F(5) - F(1) = 0$

3. $E[\underline{x}] = \int_0^1 x \cdot \frac{2}{3}(2-x)\,dx = \frac{2}{3}\int_0^1 (2x - x^2)\,dx$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}\left[1 - \frac{1}{3}\right] = \frac{4}{9}$$

$$E[\underline{x}^2] = \frac{2}{3}\int_0^1 (2x^2 - x^3)\,dx = \frac{2}{3}\left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3}\left[\frac{2}{3} - \frac{1}{4}\right] = \frac{2}{3}\left(\frac{5}{12}\right) = \frac{10}{36}$$

$$\sigma^2 = E[\underline{x}^2] - E[\underline{x}]^2 = \frac{10}{36} - \left(\frac{4}{9}\right)^2 = \frac{13}{162}$$

4. $$E[x] = \int_{-1}^{0} x(1+x)\,dx + \int_{0}^{1} x(1-x)\,dx$$

$$= \left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{1}$$

$$= -\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \underline{0}$$

$$E[x^2] = \int_{-1}^{0} x^2(1+x)\,dx + \int_{0}^{1} x^2(1-x)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_{-1}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{0}^{1}$$

$$= -\left(\frac{-1}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\sigma^2 = E[x^2] - E[x]^2 = \underline{\frac{1}{6}}$$

5. a. $$\int_0^1 \int_0^1 cxy\,dy\,dx = c\int_0^1 x\frac{y^2}{2}\Big|_0^1\,dx$$

$$= c\int_0^1 \frac{x}{2}\,dx = \frac{c}{4}x^2\Big|_0^1 = \frac{c}{4} \qquad \underline{c = 4}$$

b. $f_x(x) = \int_0^1 4xy\,dy = 4xy^2/2\big|_0^1 = \underline{2x}$

$f_y(y) = \int_0^1 4xy\,dx = \underline{2y}$

c. $$f_x(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{4xy}{2y} = \underline{2x}$$

$f_y(y|x) = \underline{2y}$

d. Yes.

6. $f(x,y) = 4xy$ on $0 \le x \le 1$, $0 \le y \le 1$

$$E[\underline{x}] = \int_0^1 \int_0^1 x 4xy \, dy \, dx = \int_0^1 \int_0^1 4x^2 y \, dy \, dx$$

$$= \int_0^1 \left[\frac{4x^2y^2}{2} \right]_0^1 dx = \int_0^1 2x^2 dx$$

$$= \frac{2}{3} x^3 \Big|_0^1 = \underline{2/3}$$

$$E[\underline{y}] = \int_0^1 \int_0^1 4xy^2 \, dy \, dx = 2/3$$

a. $E[\underline{x} - \underline{y}] = E[\underline{x}] - E[\underline{y}] = \underline{0}$

b. $E[3\underline{y}] = 3E[\underline{y}] = \underline{2}$

c. $E[\underline{x}] = 2/3$

d. $E[\underline{x} + \underline{y}] = E[\underline{x}] + E[\underline{y}] = \underline{1\frac{1}{3}}$

7. $Cov(x,y) = E[\underline{x}\underline{y}] - E[\underline{x}]E[\underline{y}]$

$E[\underline{x}\underline{y}] \neq E[\underline{x}]E[\underline{y}]$ (in general)

$$E[\underline{x}\underline{y}] = \int_0^1 \int_0^1 4x^2y^2 \, dy \, dx = 4\int_0^1 x^2 \frac{y^3}{3} \Big|_0^1 dx$$

$$= \frac{4}{3} \int_0^1 x^2 dx = \frac{4}{3} \frac{x^3}{3} \Big|_0^1 = \underline{\frac{4}{9}}$$

$$Cov(x,y) = \frac{4}{9} - \frac{2}{3} \cdot \frac{2}{3} = \underline{0}$$

$\rho = Cov(x,y) / \sigma_x \sigma_y$

$\sigma_x^2 = E[\underline{x}^2] - E[\underline{x}]^2$

$$E[\underline{x}^2] = \int_0^1 \int_0^1 4x^3 y \, dy \, dx = \int_0^1 \frac{4}{2} x^3 y^2 \Big|_0^1 dx =$$

$$\int_0^1 2x^3 dx = \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}$$

$$\sigma_x^2 = \frac{1}{2} - (2/3)^2 = \frac{1}{18}$$

$\sigma_y^2 = 1/18$ similarly

$\underline{\rho = 0}$ since $Cov = 0$.

when 2 vars are independent, $cov = \rho = 0$.

8. The moment generating function is defined as:

$$E[e^{\theta x}] = \int_{-\infty}^{\infty} f(x)\, e^{\theta x}\, dx$$

$$\therefore\ \phi(\theta) = \int_{-\infty}^{0} e^{x}\, e^{\theta x}\, dx = \int_{-\infty}^{0} e^{x(\theta+1)}\, dx$$

$$= \frac{1}{\theta+1} e^{(\theta+1)x} \Big|_{-\infty}^{0} = \frac{1}{\theta+1}\left[e^{(\theta+1)0} - 0\right]$$

$$\underline{\phi(\theta) = \frac{1}{\theta+1}}$$

The n'th moment is $\phi^{(n)}(0)$

so $\mu_1' = \phi^{(1)}(0) = \frac{d}{d\theta}\left(\frac{1}{\theta+1}\right)_{\theta\to 0}$

$$= \frac{-1}{(\theta+1)^2} = \underline{1}$$

Since μ_n' is defined as $E[\underline{x}^n]$

$$\mu_1' = E[\underline{x}] = \mu$$

$$\underline{\mu = -1}$$

note also $\mu^2 + \sigma^2 = \phi^{(2)}(0)$

$$\phi^{(2)}(\theta) = \frac{d}{d\theta} \frac{-1}{(\theta+1)^2} = \frac{2}{(\theta+1)^3} = 2 \text{ at } \theta = 0$$

so $\mu^2 + \sigma^2 = 2$ $\quad \underline{\sigma^2 = 1}$