ECE 500 - HW 2 Solutions

1. Note this is an even function so

$$b_n = 0 \quad \forall n$$

Find a_0

$$a_0 = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\,dt$$

$$= \frac{2}{T}\int_0^{T/2}\left(1-\frac{4t}{T}\right)dt + \frac{2}{T}\int_{T/2}^{T}\left(-3+{}^{4t}/_{T}\right)dt$$

$$= \frac{2}{T}\left\{t - \frac{2t^2}{T}\right\}\Big|_0^{T/2} + \frac{2}{T}\left\{-3t + \frac{2t^2}{T}\right\}\Big|_{T/2}^{T}$$

$$\frac{2}{T}\left[\left(\frac{T}{2}\right)-\left(\frac{2T^2}{4T}\right) - 0 + 0\right]$$

$$+\frac{2}{T}\left[\left(-3T+\frac{2T^2}{T}\right) - \left(-\frac{3T}{2}+\frac{2T^2}{4T}\right)\right]$$

$$= 0$$

$$a_n = \frac{2}{T}\int_0^{T/2}\left(1-\frac{4t}{T}\right)\cos n\omega_0 t\,dt$$

$$+\frac{2}{T}\int_{T/2}^{T}\left(-3+\frac{4t}{T}\right)\cos n\omega_0 t\,dt$$

Evaluate these separately:

$$\frac{2}{T}\int_0^{T/2}\left(1-\frac{4t}{T}\right)\cos n\omega_0 t\,dt = \frac{2}{T}\int_0^{T/2}\cos n\omega_0 t\,dt - \frac{2}{T}\cdot\frac{4}{T}\int_0^{T/2} t\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\,\frac{\sin n\omega_0 t}{n\omega_0}\Bigg|_0^{T/2} - \frac{8}{T^2}\int_0^{T/2} t\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\left(\frac{\sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right)}{n\omega_0} - 0\right) - \frac{8}{T^2}\int_0^{T/2} t\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\,\frac{\sin(n\pi)}{n\omega_0} - \frac{8}{T^2}\int_0^{T/2} t\cos n\omega_0 t\,dt$$

$$= -\frac{8}{T^2}\int_0^{T/2} t\cos n\omega_0 t\,dt$$

$$= \frac{-8}{T^2}\left[\frac{\cos n\omega_0 t}{n^2\omega_0^2} + \frac{t\sin n\omega_0 t}{n\omega_0}\right]_0^{T/2}$$

$$= \left(\frac{-8}{T^2}\right)\left[\frac{\cos\left(n\frac{2\pi}{T}\cdot\frac{T}{2}\right)}{n^2\omega_0^2} + \frac{\left(\frac{T}{2}\right)\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n\omega_0}\right.$$

$$\left. - \frac{\cos(0)}{n^2\omega_0^2} + 0\right]$$

$$= \left(\frac{-8}{T^2}\right)\left[\frac{1}{n^2\omega_0^2}\right]\left(\cos(n\pi) - 1\right)$$

Notice $\cos(n\pi) = -1 \quad n \text{ odd}$
$\qquad\qquad\qquad = +1 \quad n \text{ even}$

$$= \begin{cases} 0 & n \text{ even} \\ \left(\frac{-8}{T^2}\right)\left(\frac{1}{n^2\omega_0^2}\right)(-2) & n \text{ odd} \end{cases}$$

$\omega_0 = \frac{2\pi}{T}$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n^2\pi^2} & n \text{ odd} \end{cases} \longleftarrow \text{Result \# 1}$$

Second part

$$\frac{2}{T}\int_{T/2}^{T}\left(-3+\frac{4t}{T}\right)\cos n\omega_0 t\,dt$$

$$=\frac{2}{T}\int_{T/2}^{T}\underbrace{-3\cos n\omega_0 t\,dt}_{0}+\frac{2}{T}\cdot\frac{4}{T}\int_{T/2}^{T} t\cos n\omega_0 t\,dt$$

$$=\frac{8}{T^2}\left\{\frac{\cos n\omega_0 t}{n^2\omega_0^2}+\frac{t\sin n\omega_0 t}{n\omega_0}\right\}\Bigg|_{T/2}^{T}$$

$$=\frac{8}{T^2}\left\{\frac{\cos n\omega_0 T}{n^2\omega_0^2}+\frac{T\sin n\omega_0 T}{n\omega_0}\right.$$

$$\left.-\frac{\cos n\omega_0 T/2}{n^2\omega_0^2}-\frac{\frac{T}{2}\sin n\omega_0 T/2}{n\omega_0}\right\}$$

$$=\frac{8}{T^2}\left\{\frac{\cos 2\pi n}{n^2\omega_0^2}+\frac{T\sin 2\pi n}{n\omega_0}\right.$$

$$\left.-\frac{\cos \pi n}{n^2\omega_0^2}-\frac{\frac{T}{2}\sin \pi n}{n\omega_0}\right\}$$

$$=\frac{8}{T^2}\left\{\frac{1}{n^2\omega_0^2}+0-\frac{\cos\pi n}{n^2\omega_0^2}-0\right\}$$

$$=\frac{8}{T^2n^2\omega_0^2}\left\{1-\cos\pi n\right\}$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n^2\pi^2} & n \text{ odd} \end{cases} \longleftarrow \quad \text{RESULT \#2}$$

so for n even $a_n = 0$

n odd $a_n = \dfrac{8}{n^2\pi^2}$

(add the two results we've found so far)

Therefore, the Fourier Series representation is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$= a_1 \cos 1 \cdot \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + a_4 \cos 4\omega_0 t + a_5 \cos 5\omega_0 t + a_6 \cos 6\omega_0 t + \ldots$$

but $a_n = 0$ when n is even

$$= a_1 \cos 1\omega_0 t + a_3 \cos 3\omega_0 t + a_5 \cos 5\omega_0 t + a_7 \cos 7\omega_0 t + \ldots$$

$$= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{\pi^2 n^2} \cos \frac{2\pi}{T} n t$$

$$= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2\pi}{T} (2n-1) t$$

since $(2n-1)$ is always odd.

2a) $x(t) = e^{-\alpha|t|}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

when $t > 0$ $\quad e^{-\alpha|t|} = e^{-\alpha t}$

$t < 0$ $\quad e^{-\alpha|t|} = e^{\alpha t}$

so
$$X(\omega) = \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{-(j\omega-\alpha)t}\, dt + \int_{0}^{\infty} e^{-(j\omega+\alpha)t}\, dt$$

$$= \left.\frac{e^{-(j\omega-\alpha)t}}{-(j\omega-\alpha)}\right|_{-\infty}^{0} + \left.\frac{e^{-(j\omega+\alpha)t}}{-(j\omega+\alpha)}\right|_{0}^{\infty}$$

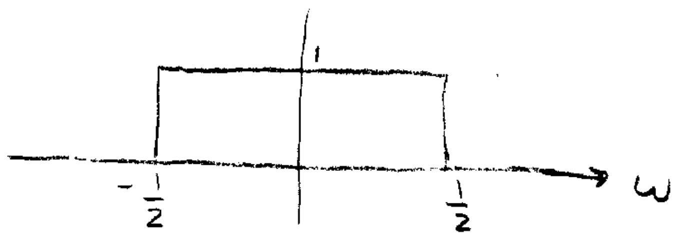
$$= \left[\frac{1}{-(j\omega-\alpha)} - 0\right] + \left[0 - \frac{1}{-(j\omega+\alpha)}\right]$$

$$= \frac{1}{(\alpha - j\omega)} + \frac{1}{(\alpha + j\omega)}$$

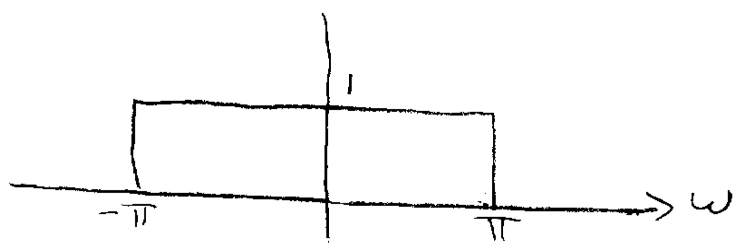
$$= \frac{(\alpha - j\omega) + (\alpha + j\omega)}{(\alpha - j\omega)(\alpha + j\omega)} = \underline{\frac{2\alpha}{\alpha^2 + \omega^2}}$$

2b. $X(\omega) = \text{rect}\left(\omega/2\pi\right)$

recall, $\text{rect}(\omega)$ is



So $\text{rect}\left(\omega/2\pi\right)$ is



$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \cdot e^{j\omega t}\, d\omega$$

$$= \left(\frac{1}{2\pi}\right) \frac{e^{j\omega t}}{jt}\Bigg|_{-\pi}^{\pi}$$

2b. Cont

$$= \left(\frac{1}{2\pi}\right)\left[\frac{e^{j\pi t} - e^{-j\pi t}}{jt}\right]$$

$$= \left(\frac{1}{t\pi}\right)\left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right)$$

From the handout

$$= \left(\frac{1}{t\pi}\right)\left(\sin \pi t\right)$$

$$= \frac{\sin \pi t}{\pi t} = \operatorname{Sinc} t$$