1a

1-1

$$\int_{-3}^{3} \alpha(16-x^2)\,dx = \left(16\alpha x - \alpha\frac{x^3}{3}\right)\Big|_{-3}^{3} =$$

$$\left(48\alpha - \frac{27}{3}\alpha\right) - \left(-48\alpha + \frac{27}{3}\alpha\right) = 96\alpha - 18\alpha = 78\alpha$$

we know $\int_{-\infty}^{\infty} f(x)\,dx = 1$, so $\underline{\alpha = 1/78}$

$$F(x) = \int_{-3}^{x} \tfrac{1}{78}(16-x^2)\,dx = \frac{16}{78}x - \frac{x^3}{3\cdot 78}\Bigg|_{-3}^{x} \quad \text{on } -3 \le x \le 3$$

$$F(x) = \begin{cases} 0 & x < -3 \\ \left(\frac{16}{78}x - \frac{x^3}{234}\right) - \left(-3\cdot\frac{16}{78} + \frac{27}{3\cdot 78}\right) & -3 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

$$= \begin{cases} 0 & x < -3 \\ \frac{16}{78}x - \frac{x^3}{234} + 0.5 & -3 \le x < 3 \\ 1 & x > 3 \end{cases}$$

b. $$E[\underline{x}] = \int_{-3}^{3} \tfrac{1}{78}\, x(16-x^2)\,dx = \frac{1}{78}\left[8x^2 - \frac{x^4}{4}\right]_{-3}^{3}$$

$$= \frac{1}{78}\left[\left(8\cdot 9 - \frac{81}{4}\right) - \left(8\cdot 9 - \frac{81}{4}\right)\right] = \underline{0 \text{ minutes}}$$

$$E[\underline{x}^2] = \int_{-3}^{3} \frac{1}{78} x^2(16-x^2)\,dx = \frac{1}{78}\left[16/3\, x^3 - \frac{x^5}{5}\right]_{-3}^{3}$$

$$= \frac{1}{78}\left[\left(\frac{16}{3}\cdot 27 - \frac{243}{5}\right) - \left(\frac{16}{3}\cdot -27 + \frac{243}{5}\right)\right] = \frac{190.80}{78}$$

$$\sigma^2 = E[\underline{x}^2] - E[\underline{x}]^2 = \underline{2.446 \text{ minutes}^2}$$

1c. Rewrite everything in seconds

$$f(y) = \begin{cases} \beta\,(16 \cdot 60^2 - y^2) & -180 \le y \le 180 \\ 0 & \text{o w} \end{cases}$$

$$\beta \int_{-180}^{180} (57600 - y^2)\,dy = \beta \left(57600y - y^3/3\right)\Big|_{-180}^{180}$$

$$= \beta \{16848000\}, \quad \beta = \frac{1}{16{,}848{,}000}$$

$$E[y^2] = \int_{-180}^{180} \beta y^2 (57600 - y^2)\,dy = \beta \left[\frac{57600 y^3}{3} - \frac{y^5}{5}\right]\Big|_{-180}^{180}$$

$$= \beta\left[1.4837 \times 10^{11}\right] = \underline{8.8062 \times 10^3 \text{ seconds}}$$

$E[y] = 0$

so $\underline{\sigma^2 = 8806.2 \text{ seconds}^2}$

Note we can get this via $\sigma^2 = 2.446 \text{ minutes}^2$
$= 2.446 \times 60 \times 60 \text{ seconds}^2$

2a. $\displaystyle\int_0^2\int_0^2 f(x,y)\,dy\,dx$

$$= \underbrace{\int_0^1\int_0^1 xy\,dy\,dx}_{A} + \underbrace{\int_1^2\int_0^1 (2-x)y\,dy\,dx}_{B} + \underbrace{\int_0^1\int_1^2 x(2-y)\,dy\,dx}_{C}$$

$$+ \underbrace{\int_1^2\int_1^2 (2-x)(2-y)\,dy\,dx}_{D}$$

A: $\displaystyle\int_0^1\int_0^1 xy\,dy\,dx = \int_0^1 \frac{xy^2}{2}\Big|_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4}$

B: $\displaystyle\int_1^2\int_0^1 (2-x)y\,dy\,dx = \int_1^2 \left(y^2 - \frac{xy^2}{2}\right)\Big|_0^1 dx = \int_1^2 \left(1 - \frac{x}{2}\right)dx$

$$= x - \frac{x^2}{4}\Big|_1^2 = (2-1) - \left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

C: $\displaystyle\int_0^1\int_1^2 x(2-y)\,dy\,dx = \int_0^1 \left(2xy - \frac{y^2 x}{2}\right)\Big|_1^2 dx$

$$= \int_0^1 \left(2x - \frac{3x}{2}\right)dx = \cancel{x^2}\; x^2 - \frac{3x^2}{4}\Big|_0^1 = \frac{1}{4}$$

D: $\displaystyle\int_1^2\int_1^2 (4 - 2x - 2y + xy)\,dy\,dx = \int_1^2 \left(4y - 2xy - y^2 + \frac{xy^2}{2}\right)\Big|_1^2 dx$

$$= \int_1^2 \left(4 - 2x - 3 + \frac{3}{2}x\right)dx = 4x - x^2 - 3x + \frac{3x^2}{4}\Big|_1^2$$

$$= 4 - 3 - 3 + \frac{3}{4}(3) = \frac{1}{4}$$

$A + B + C + D = \underline{1}$

3a. First find C:

$$\int_0^\infty \int_0^\infty C e^{-(x+y)}\, dy\, dx = \int_0^\infty\int_0^\infty C e^{-x} e^{-y}\, dy\, dx$$

$$\int_0^\infty C e^{-x}(-e^{-y})\Big|_0^\infty dx = \int_0^\infty C e^{-x}\, dx$$

$$= -C e^{-x}\Big|_0^\infty = C \quad \text{so} \quad \underline{C = 1}$$

$$E[x] = \int_0^\infty \int_0^\infty x e^{-x} e^{-y}\, dy\, dx =$$

$$\int_0^\infty \left[-x e^{-x} e^{-y}\right]_0^\infty dx = \int_0^\infty x e^{-x}\, dx$$

$$= \frac{e^{-x}}{-1}\left[x+1\right]\Big|_0^\infty = \boxed{1 = E[x]}$$

$$E[y] = \int_0^\infty \int_0^\infty y e^{-x} e^{-y}\, dy\, dx =$$

$$= \int_0^\infty e^{-x}\left[\frac{e^{-y}}{-1}(y+1)\right]_0^\infty dx$$

$$= \int_0^\infty e^{-x}[1]\, dx = -e^{-x}\Big|_0^\infty = \boxed{1 = E[y]}$$

$$E[x+y] = E[x] + E[y] = \boxed{2 = E[x+y]}$$

$$E[xy] = \int_0^\infty \int_0^\infty x y e^{-x} e^{-y}\, dy\, dx$$

$$= \int_0^\infty x e^{-x} \left[\frac{e^{-y}}{-1} (y+1) \right]_0^\infty dx$$

$$= \int_0^\infty x e^{-x} [1] dx = \frac{e^{-x}}{-1}(x+1) \Big|_0^\infty = 1$$

$$\boxed{E[\underline{x}\,\underline{y}] = 1}$$

b. $$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^{\infty} e^{-x} e^{-y}\, dy$$

$$= e^{-x}\left[-e^{-y}\right]_0^\infty = \underline{e^{-x}} \quad \text{on } 0 \le x \le \infty$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^{\infty} e^{-x} e^{-y}\, dx$$

$$= e^{-y}\left[-e^{-x}\right]_0^\infty = \underline{e^{-y}} \quad \text{on } 0 \le y \le \infty$$

c. $$f_x(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{e^{-x}e^{-y}}{e^{-y}} = \underline{e^{-x} \quad \text{on } 0 \le x \le \infty}$$

$$f_y(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{e^{-x}e^{-y}}{e^{-x}} = \underline{e^{-y} \quad \text{on } 0 \le y \le \infty}$$

4a. $$\int_0^1\int_0^2\int_0^\infty c(y_1+y_2)e^{-y_3}\,dy_3\,dy_2\,dy_1 = 1$$

$$= \int_0^1\int_0^2 \left[c(y_1+y_2)[-e^{-y_3}]\Big|_0^\infty \right] dy_2\,dy_1$$

$$= \int_0^1\int_0^2 c(y_1+y_2)\,dy_2\,dy_1 = \int_0^1 c\left(y_1y_2+\frac{y_2^2}{2}\right)\Big|_0^2 dy_1$$

$$= \int_0^1 c\left[2y_1+2\right]dy_1 = c\left[y_1^2+2y_1\right]_0^1 = 3c$$

$$\therefore \boxed{c = 1/3}$$

b. $$F_{y_1}(y_1) = \int_0^2\int_0^\infty \frac{1}{3}(y_1+y_2)e^{-y_3}\,dy_3\,dy_2$$

$$= \int_0^2 \frac{1}{3}(y_1+y_2)(-e^{-y_3})\Big|_0^\infty dy_2$$

$$= \int_0^2 \frac{1}{3}(y_1+y_2)\,dy_2 = \frac{1}{3}\cdot y_1y_2+\frac{1}{3}\frac{y_2^2}{2}\Big|_0^2$$

$$= \frac{y_1}{3}\cdot 2 + \frac{1}{3}\cdot 2 = \underline{\frac{2}{3}(y_1+1)} \qquad 0\le y_1\le 1$$

b. $f_{y_2}(y_2) = \int_0^1 \int_0^\infty \frac{1}{3}(y_1 + y_2) e^{-y_3}\, dy_3\, dy_1$

$= \int_0^1 \frac{1}{3}(y_1 + y_2)\left(-e^{-y_3}\right)\Big|_0^\infty \, dy_1$

$= \frac{1}{3}\int_0^1 (y_1 + y_2)\, dy_1 \; = \frac{1}{3}\left(\frac{y_1^2}{2} + y_2 y_1\right)\Big|_0^1$

$= \underline{\frac{1}{3}\left[\frac{1}{2} + y_2\right]} \qquad 0 \le y_2 \le 2$

$f_{y_3}(y_3) = \int_0^1 \int_0^2 \frac{1}{3}(y_1 + y_2) e^{-y_3}\, dy_2\, dy_1$

$= \int_0^1 \left(\frac{e^{-y_3}}{3}\right)\left(y_1 y_2 + \frac{y_2^2}{2}\right)\Big|_0^2 \, dy_1$

$y_1\Big|_0$ $\qquad\qquad$ $\cancel{y_1}\Big)\Big|_0^1 e^{-y_3}$

$=$

$= \frac{e^{-y_3}}{3}\int_0^1 [2y_1 + 2]\, dy_1 = \frac{e^{-y_3}}{3}\left[y_1^2 + 2y_1\right]_0^1$

$= \frac{e^{-y_3}}{3}[3] = \underline{e^{-y_3}} \qquad y_3 > 0$

b. The variables are not independent since $f(y_1, y_2, y_3) \neq f_{y_1}(y_1) f_{y_2}(y_2) f_{y_3}(y_3)$

c. See part b.

5. $f(x,y) = \begin{cases} cxy & 0 \le x \le 1 \quad 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$

$$\int_0^1 \int_0^1 cxy \, dy \, dx = c \int_0^1 x \left. \frac{y^2}{2} \right|_0^1 dx$$

$$= c \int_0^1 \frac{x}{2} dx = \left. c \frac{x^2}{4} \right|_0^1 = \frac{c}{4} \qquad \boxed{c = 4}$$

a. $Cov(x,y) = E[\underline{x}\underline{y}] - E[\underline{x}]E[\underline{y}]$

$$E[\underline{x}\underline{y}] = \int_0^1 \int_0^1 4x^2y^2 \, dy \, dx$$

$$= \int_0^1 4x^2 \left. \frac{y^3}{3} \right|_0^1 dx = \int_0^1 \frac{4}{3} x^2 \, dx$$

$$= \left. \frac{4x^3}{9} \right|_0^1 = \frac{4}{9}$$

$$E[\underline{x}] = \int_0^1 \int_0^1 4x^2 y \, dy \, dx = \int_0^1 4x^2 \left. \frac{y^2}{2} \right|_0^1 dx$$

$$= \int_0^1 2x^2 \, dx = \left. \frac{2}{3} x^3 \right|_0^1 = \underline{2/3}$$

$$E[\underline{y}] = \int_0^1 \int_0^1 4xy^2 \, dy \, dx = \int_0^1 \left. \frac{4}{3} xy^3 \right|_0^1 dx$$

$$= \int_0^1 \frac{4}{3} x \, dx = \left. \frac{4}{6} x^2 \right|_0^1 = \underline{\frac{2}{3}}$$

$$Cov(x,y) = \frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \underline{0}$$

b. Cannot assume independence just because Covariance $= 0$.

$$f_x(x) = \int_0^1 4xy\,dy = 2xy^2\Big|_0^1 = 2x \qquad 0 \le x \le 1$$

$$f_y(y) = \int_0^1 4xy\,dx = 2yx^2\Big|_0^1 = 2y \qquad 0 \le y \le 1$$

$f(x,y) = f_x(x)\,f_y(y)$ $\therefore$ they are independent

c. Use the marginals from above:

$$P\{.5 \le x \le .9\} = \int_{.5}^{.9} f_x(x)\,dx$$

$$= \int_{.5}^{.9} 2x\,dx = x^2\Big|_{.5}^{.9}$$

$$= .9^2 - .5^2 = 0.56$$

d. We showed in b that x & y are independent, so knowledge of y doesn't change anything about x,

ans $\underline{0.56}$

In general, we know that

$$f(x|y) = \frac{f(x,y)}{f_y(y)}, \quad \text{or}$$

$$F(x|y) = \frac{\int\int f(x,y)}{\int f_y(y)}$$

in our case

$$= \frac{\int_{.5}^{.9}\int_{.5}^{1} 4xy\, dy\, dx}{\int_{.5}^{1} 2y\, dy}$$

$$= \frac{\int_{.5}^{.9} 2xy^2\Big|_{.5}^{1} dx}{y^2\Big|_{.5}^{1}} = \frac{\int_{.5}^{.9} 2x\, dx\,(1-.5^2)}{(1-.5^2)}$$

$$= \int_{.5}^{.9} 2x\, dx = x^2\Big|_{.5}^{.9} = \underline{0.56}$$

e. From earlier, we know

$\underline{\mu_x = 2/3}$

$\underline{\mu_y = 2/3}$

$$E[(x-\mu)^2] = E[x^2] - E[x]^2$$

$$E[x^2] = \int_0^1 \int_0^1 4x^3 y \, dy \, dx = \int_0^1 2x^3 y^2 \Big|_0^1 dx$$

$$= \int_0^1 2x^3 \, dx = \tfrac{1}{2} x^4 \Big|_0^1 = \tfrac{1}{2}$$

so $\sigma_x^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9-8}{18} = 1/18$

$\underline{\sigma_x = \sqrt{1/18}}$

Similarly $\underline{\sigma_y = \sqrt{1/18}}$