KEY

1. $x(t) = 10t^2$ on $0,1$ $\quad T = 1$, $\omega_0 = \frac{2\pi}{T} = 2\pi$

<u>Exponential</u> $\quad C_N = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\, e^{-jn\omega_0 t}\, dt$

$$C_N = \int_0^1 x(t)\, e^{-j2\pi n t}\, dt$$

$$C_0 = \int_0^1 (10t^2)\,1\, dt = 10t^3/3 = 10/3$$

$$C_n = \int_0^1 10t^2\, e^{-j2\pi n t}\, dt$$

Using tables, we know

$$\int t^2 e^{at} = \frac{e^{at}}{a}\left(t^2 - \frac{2t}{a} + \frac{2}{a^2}\right)$$

here $a = -j2\pi n$, so

$$C_N = 10\left[\frac{e^{-j2\pi n t}}{-j2\pi n}\left(t^2 - \frac{2t}{-j2\pi n} + \frac{2}{(-j2\pi n)^2}\right)\right]\Bigg|_0^1$$

$$= 10\left[\frac{e^{-j2\pi n t}}{-j2\pi n}\left(t^2 + \frac{2t}{j2\pi n} - \frac{2}{4\pi^2 n^2}\right)\right]_0^1$$

$$= 10\left[\frac{e^{-j2\pi n}}{-j2\pi n}\right]\left\{1 + \frac{2}{j2\pi n} - \frac{2}{4\pi^2 n^2}\right\} -$$

$$\left[\frac{1}{-j2\pi n}\right]\left\{0 + 0 - \frac{2}{4\pi^2 n^2}\right\}\Bigg]$$

$$= 10\left[\frac{1}{-j2\pi n}\right]\left[1 + \frac{2}{j2\pi n}\right]$$

$$= -10\left[\frac{1}{j2\pi n}\right]\left[\frac{j2\pi n + 2}{j2\pi n}\right]$$

$$= \frac{-10 \cdot (j2\pi n + 2)}{j^2 4\pi^2 n^2}$$

$$= \frac{10 \cdot (j2\pi n + 2)}{4\pi^2 n^2}$$

note $e^{-j2\pi n} = \cos(-2\pi n) + j\sin(-2\pi n)$

$= 1$ always

since $\cos(2\pi) = \cos(4\pi) \ldots = 1$
$\sin(2\pi) = \sin(4\pi) \ldots = 0$

$$C_n = \frac{10(j2\pi n + 2)}{4\pi^2 n^2} = \frac{5(j\pi n + 1)}{\pi^2 n^2}$$

and $\therefore$

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\boxed{= \frac{10}{3} + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{5(j\pi n + 1)}{\pi^2 n^2} e^{j2\pi n t}}$$

Now find the trigonometric series:

<u>Short cut</u>

From before,

$$x(t) = \frac{10}{3} + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{5(j\pi n + 1)}{\pi^2 n^2} e^{j2\pi n t}$$

$$= \frac{10}{3} + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{5(j\pi n + 1)}{\pi^2 n^2} \left[\cos 2\pi n t + j \sin 2\pi n t\right]$$

$$= \frac{10}{3} + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{5(j\pi n + 1)}{\pi^2 n^2} \cos 2\pi n t + \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \frac{5j(j\pi n + 1)}{\pi^2 n^2} \sin 2\pi n t$$

$$= \frac{10}{3} + \sum_{n=1}^{\infty} \left(\frac{5(j\pi n+1)}{\pi^2 n^2} + \frac{5(-j\pi n+1)}{\pi^2(-n)^2} \right) \cos 2\pi n t$$

$$+ \sum_{n=1}^{\infty} \left(\frac{5j(j\pi n+1)}{\pi^2 n^2} - \frac{5j(-j\pi n+1)}{\pi^2(-n)^2} \right) \sin 2\pi n t$$

$$\boxed{x(t) = \frac{10}{3} + \sum_{n=1}^{\infty} \left(\frac{10}{\pi^2 n^2} \cos 2\pi n t - \frac{10}{\pi n} \sin 2\pi n t \right)}$$

<u>Long way</u>

$$a_0 = 2\int_0^1 10t^2\, dt = \frac{20}{3}, \therefore \text{ first term is } \frac{10}{3}$$

$$a_n = 2\int_0^1 10t^2 \cos 2\pi n t\, dt, \quad b_n = 2\int_0^1 10t^2 \sin 2\pi n t\, dt$$

using $\displaystyle\int t^2 \sin at\, dt = \frac{2t}{a^2}\sin at + \left(\frac{2}{a^3} - \frac{t^2}{a}\right)\cos at$

with $a = 2\pi n$,

$$b_n = 20 \left[\frac{2t}{(2\pi n)^2} \sin 2\pi n t + \left(\frac{2}{(2\pi n)^3} - \frac{t^2}{2\pi n} \right) \cos 2\pi n t \right]_0^1$$

$$= 20\left\{\left[\frac{2}{(2\pi n)^2}\sin 2\pi n + \left(\frac{2}{(2\pi n)^3} - \frac{1}{2\pi n}\right)\cos 2\pi n\right] - \right.$$

$$\left.\left[0 + \left(\frac{2}{(2\pi n)^3} - 0\right)\cos 0\right]\right\}$$

$$= 20\left[\frac{2 \cdot 0}{(2\pi n)^2} + \left(\frac{2}{(2\pi n)^3} - \frac{1}{2\pi n}\right)(1) - \right.$$

$$\left.\left(\frac{2}{(2\pi n)^3}\right)(1)\right]$$

Since $\cos 2\pi n = 1 \quad \forall n$
$\sin 2\pi n = 0 \quad \forall n$

$$= 20 \cdot \left(\frac{-1}{2\pi n}\right) = \frac{-10}{\pi n}$$

(note this is the same thing we found on 1-4)

$$a_n = 2\int_0^1 10t^2 \cos 2\pi n t \, dt$$

using $\int t^2 \cos at \, dt = \frac{2t}{a^2}\cos at + \left(\frac{t^2}{a} - \frac{2}{a^3}\right)\sin at,$

$$a_n = 20\left[\frac{2t}{(2\pi n)^2}\cos 2\pi n t + \left(\frac{t^2}{2\pi n} - \frac{2}{(2\pi n)^3}\right)\sin 2\pi n t\right]\Bigg|_0^1$$

$$= 20\left[\left\{\frac{2}{(2\pi n)^2}\cos 2\pi n + \left(\frac{1}{2\pi n} - \frac{2}{(2\pi n)^3}\right)\sin 2\pi n\right\}\right.$$

$$\left. - \left\{ 0 + 0 - 0 \right\}\right]$$

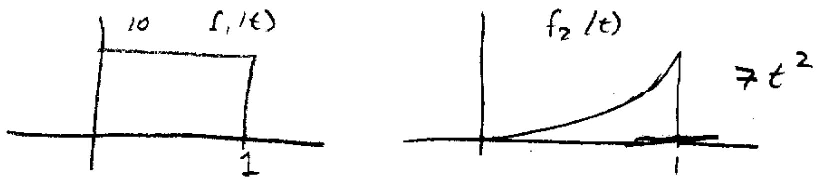
$$= 20\left[\frac{2}{(2\pi n)^2} + (\;\cdot\;)0\right] = \frac{40}{4\pi^2 n^2} = \frac{10}{\pi^2 n^2}$$

Same thing again! since $\cos 2\pi n = 1$
$\sin 2\pi n = 0$

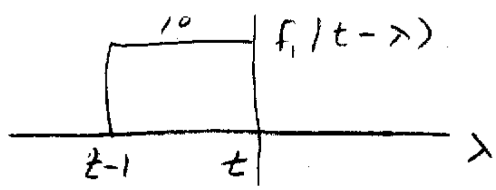
and so

$$\boxed{x(t) = \frac{10}{3} + \sum_{n=1}^{\infty}\left(\frac{10}{\pi^2 n^2}\cos 2\pi n t - \frac{10}{\pi n}\sin 2\pi n t\right)}$$

2 Convolve the following



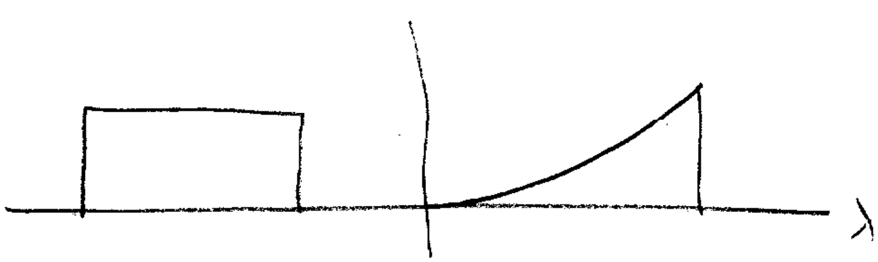
'Fold' f_1 :



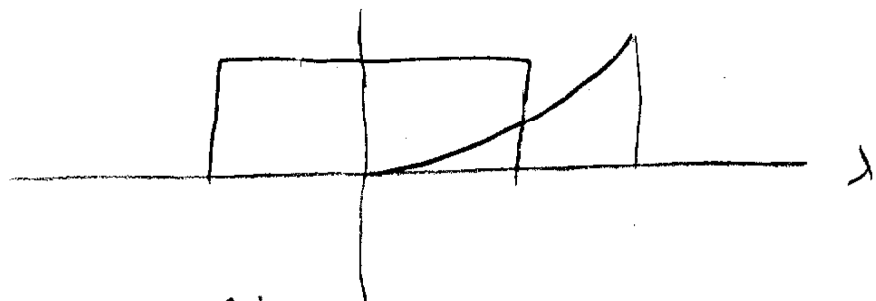
The four distinct regions are

R_1: $t < 0$ No overlap
R_2: $0 \leq t < 1$ Some overlap
R_3: $1 \leq t < 2$ Some overlap
R_4: $t \geq 2$ No overlap

R_1: $f_1 * f_2 = 0$



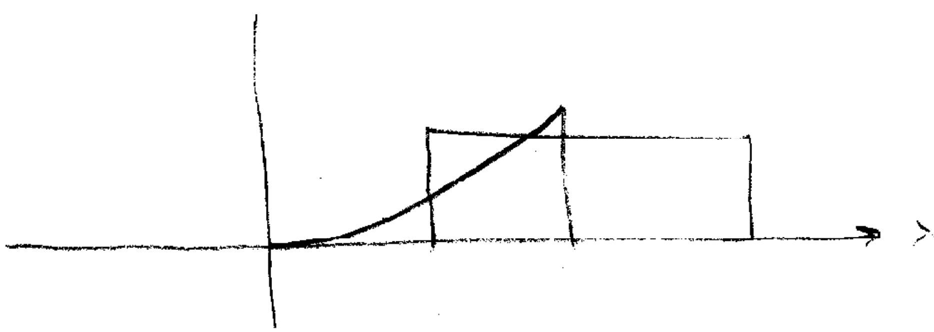
R_2 : $0 < t < 1$



$$f_1 * f_2 = \int_0^t 10 \cdot 7\lambda^2 \, d\lambda = \frac{70}{3}\lambda^3 \Big|_0^t$$

$$= 70/3 \, t^3$$

R_3. $1 < t < 2$

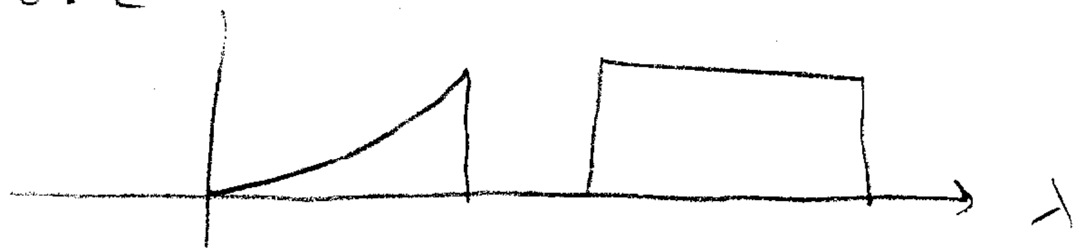


$$f_1 * f_2 = \int_{t-1}^{1} 10 \cdot 7\lambda^2 \, d\lambda = \frac{70}{3}\lambda^3 \Big|_{t-1}^{1}$$

$$= \left(\frac{70}{3} - \frac{70}{3}(t-1)^3\right)$$

$$= \frac{70}{3}\left[-t^3 + 3t^2 - 3t + 2\right]$$

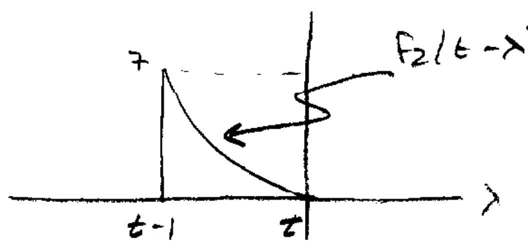
R4: $t > 2$

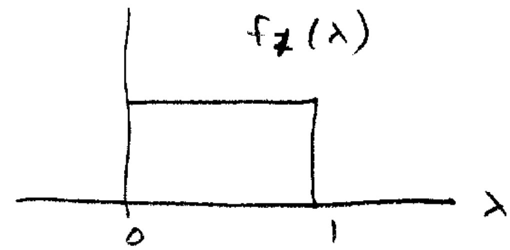


$f_1 * f_2 = 0$

$$f_1 * f_2 = \begin{cases} 0 & t < 0 \\ \frac{70}{3} t^3 & 0 \le t < 1 \\ \frac{70}{3}\left[-t^3 + 3t^2 - 3t + 2\right] & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

The other way; 'fold' f_2





$$f_2(t-\lambda) = 7(t-\lambda)^2 = 7(t^2+\lambda^2-2t\lambda)$$

Once again, four Regions

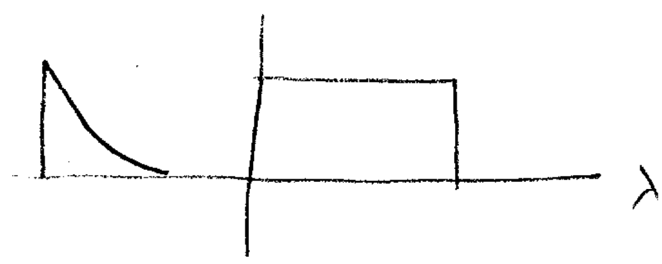
R_1 : $t < 0$

R_2 : $0 \leq t < 1$

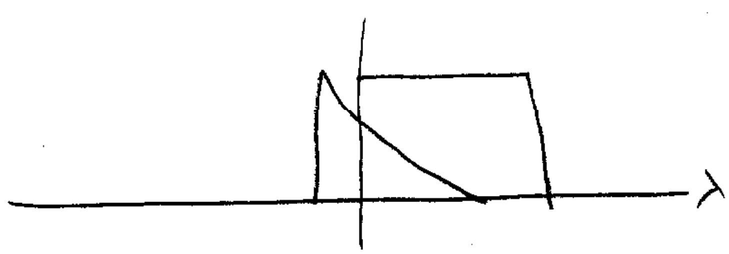
R_3 : $1 \leq t < 2$

R_4 : $t > 2$

R_1 : $f_1 * f_2 = 0$



R_2:

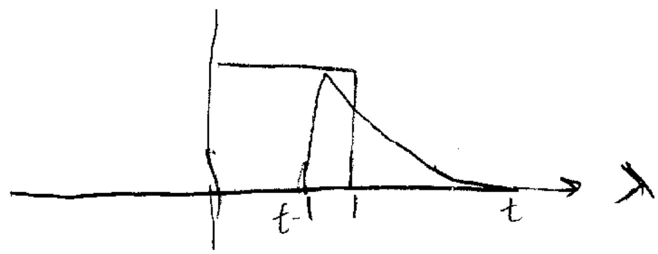


$$f_1 * f_2 = \int_0^t 70 \cdot \left[t^2+\lambda^2-2t\lambda\right] d\lambda$$

$$= 70\left[t^2\lambda + \frac{\lambda^3}{3} - \lambda^2 t\right]_0^t$$

$$= 70\left[t^3 + t^3/3 - t^3\right] = \frac{70}{3}t^3$$

R_3:



$$f_1 * f_2 = \int_{t-1}^{1} 70\left(t^2 + \lambda^2 - 2t\lambda\right)d\lambda$$

$$= 70\left[t^2\lambda + \frac{\lambda^3}{3} - t\lambda^2\right]_{t-1}^{1}$$

$$= 70\left[\left\{t^2 + \frac{1}{3} - t\right\} - \left\{(t-1)t^2 + \frac{(t-1)^3}{3} - t(t-1)^2\right\}\right]$$

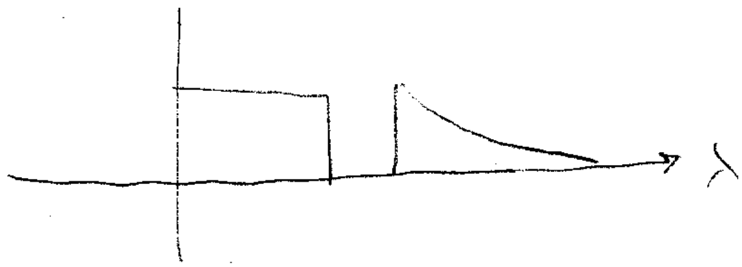
$-2t+1$
$t-1$
$2t^2+t$
t^2+2t-1
$3t^2+3t-1$

$$= 70\left[t^2 + \frac{1}{3} - t - t^3 + t^2 - \frac{(t-1)^3}{3} + t^3 - 2t^2 + t\right]$$

$$= 70\left[\frac{1}{3} - \frac{t^3}{3} + \frac{3t^2}{3} - \frac{3t}{3} + \frac{1}{3}\right]$$

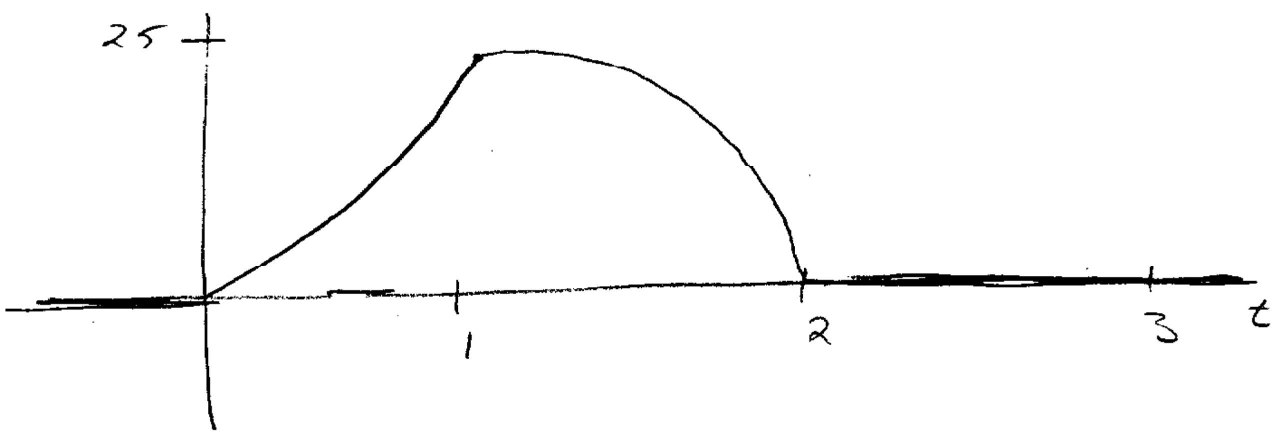
$$= \frac{70}{3}\left[-t^3 + 3t^2 - 3t + 2\right]$$

R_4:



$$f_1 * f_2 = \begin{cases} 0 & t < 0 \\ \frac{70}{3} t^3 & 0 \le t < 1 \\ \frac{70}{3}\left[-t^3 + 3t^2 - 3t + 2\right] & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$$

The convolution looks like



3. a) $H(\omega) = \dfrac{1}{2 + j\omega} = \boxed{\dfrac{V_{out}(\omega)}{V_{IN}(\omega)}}$

Long way:

$-V_{IN} + V_{1\Omega} + V_{1H} + V_{1\Omega} = 0$

$-V_{IN} + iR + L\,di/dt + iR = 0$

$-V_{IN}(\omega) + I(\omega)R + j\omega L I(\omega) + I(\omega)R = 0$

$I(\omega) = \dfrac{V_{IN}(\omega)}{2R + j\omega L} = \dfrac{V_{IN}(\omega)}{2 + j\omega}$

$V_{out}(\omega) = 1\Omega \cdot I(\omega) = \boxed{\dfrac{V_{IN}(\omega)}{2 + j\omega}}$

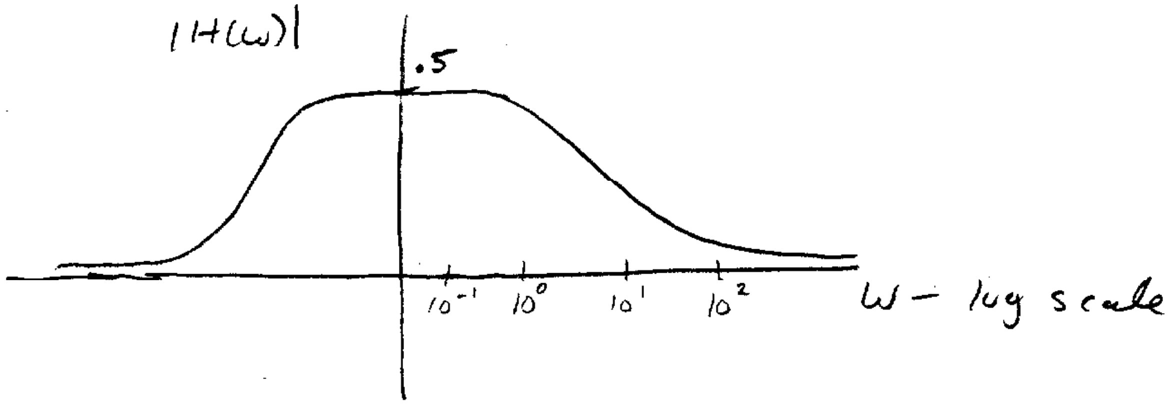
b) if $V_{in}(t) = \delta(t)$
$V_{IN}(\omega) = 1$

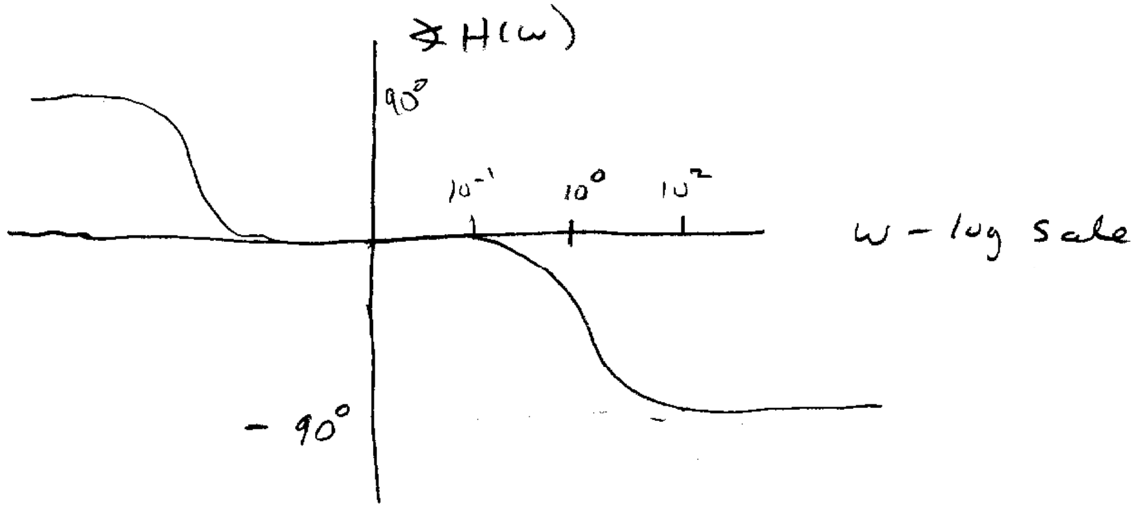
$Y(\omega) = \dfrac{1}{2 + j\omega}$ and $\boxed{h(t) = e^{-2t}u(t)}$

c) $H(\omega) = \dfrac{1}{2+j\omega}$

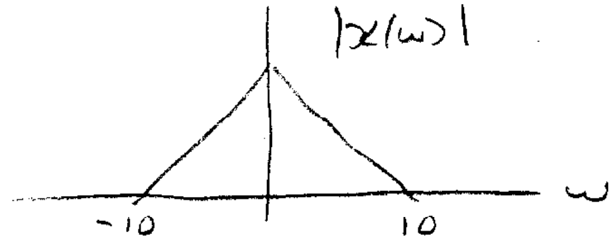
$$|H(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$

$$\measuredangle H(\omega) = -\tan^{-1}(\omega/2)$$





4 a)



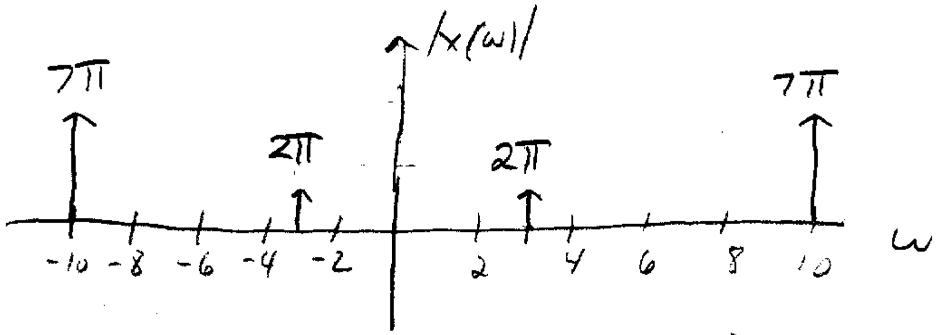
$\omega_c = 10$ rad/sec

ω_s must be 20 rad/sec or more

$$f_s = \frac{1}{2\pi}\,\omega_s = \frac{1}{2\pi}\cdot 20 = \boxed{10/\pi \text{ hertz}}$$

b) $x(t) = 2\cos 3t + 7\sin 10t$

$$X(\omega) = 2\left[\pi\cdot\delta(\omega-3) + \pi\cdot\delta(\omega+3)\right] + 7\left[j\pi\cdot\delta(\omega+10) - j\pi\,\delta(\omega-10)\right]$$



Band limited to $\omega_c = 10$ rad/sec again!

$\boxed{f_s = 10/\pi \text{ hertz}}$

5. $h(t) = te^{-2t}u(t)$

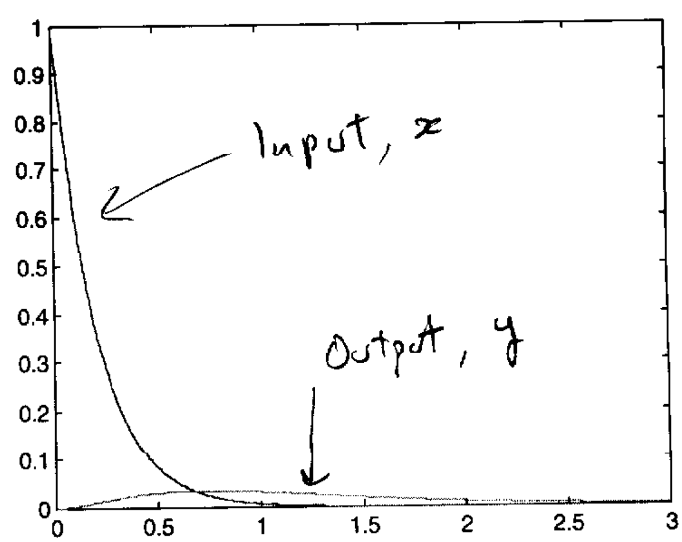
$$H(s) = \frac{1}{(s+2)^2}$$

$x(t) = e^{-5t}u(t)$

$$X(s) = \frac{1}{s+5}$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \mathcal{L}^{-1}[X(s)H(s)] \\ &= \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2(s+5)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1/3}{(s+2)^2} + \frac{-1/9}{(s+2)} + \frac{1/9}{(s+5)}\right] \end{aligned}$$

$$= \left(\frac{1}{3}te^{-2t} - \frac{1}{9}e^{-2t} + \frac{1}{9}e^{-5t}\right)u(t)$$



6.

$$\theta(s) = \frac{1}{s^2(s^2+10s+50)}$$

$$= \frac{1/50}{s^2} + \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+10s+50}$$

$$= \frac{\frac{1}{50}(s^2+10s+50) + K_1 s(s^2+10s+50) + K_2 s^3 + K_3 s^2}{s^2(s^2+10s+50)}$$

$$= \frac{s^3\left[K_2+K_1\right] + s^2\left[10K_1+K_3+\frac{1}{50}\right] + s\left[\frac{10}{50}+50K_1\right] + 1}{s^2(s^2+10s+50)}$$

$$K_1 = \frac{-10}{2500} = \underline{\frac{-1}{250}}$$

$$\underline{K_2 = \frac{1}{250}}$$

$$10K_1 + K_3 + \frac{1}{50} = \frac{-10}{250} + K_3 + \frac{5}{250}$$

$$\underline{K_3 = 5/250}$$

$$\Theta(s) = \frac{1}{250}\left[\frac{5}{s^2} - \frac{1}{s} + \frac{s+5}{s^2+10s+50}\right]$$

$$= \frac{1}{250}\left[\frac{5}{s^2} - \frac{1}{s} + \frac{(s+5)}{(s+5)^2+5^2}\right]$$

$$\theta(t) = \frac{1}{250}\left[5t - 1 + \cos 5t\, e^{-5t}\right] u(t)$$