PARTIAL FRACTION EXPANSION

Given a ratio of polynomials, $F(s) = \frac{N(s)}{D(s)}$ with degree of $N(s)$ $<$ degree of $D(s)$, find its partial fraction expansion.

Case 1: Roots of the Denominator $D(s)$ are Real and Distinct

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$

$$= \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)}$$

$$K_m = \frac{N(s)(s+p_m)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$

Case 2: Roots of the Denominator $D(s)$ are Real and repeated

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= \frac{K_{11}}{(s+p_1)^r} + \frac{K_{21}}{(s+p1)^{r-1}} + \cdots + \frac{K_{r1}}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)}$$

$$K_{i1} = \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[\frac{N(s)(s+p_1)^r}{(s+p_1)^r(s+p_2)\cdots(s+p_n)} \right], \quad i = 1, \cdots, r$$

$K_2, \cdots, K_n$, are found as before.

Case 3: Roots of the Denominator $D(s)$ are Complex or Imaginary

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s^2 + cs + d)}$$

$$= \frac{K_1}{(s + p_1)} + \frac{K_2 s + K_3}{(s^2 + cs + d)}$$

Find K_1 as before, then equate the numerator polynomials on either side of the equality:

$$N(s) = K_1(s^2 + cs + d) + (K_2 s + K_3)(s + p_1)$$

Next determine K_2 and K_3 by equating coefficients of like terms on either side of the equation. To find the inverse transform note that:

$$\frac{K_2 s + K_3}{(s^2 + cs + d)} = \frac{K_2 s + K_3}{\left[\left(s + \frac{c}{2}\right)^2 + \left(\sqrt{d - \frac{c^2}{4}}\right)^2\right]} = \frac{K_2 s + K_3}{\left[(s + a)^2 + b^2\right]}$$

where $a = \frac{c}{2}$, and $b = \sqrt{d - \frac{c^2}{4}}$ or $(a^2 + b^2) = d$

Now $$\frac{K_2 s + K_3}{\left[(s + a)^2 + b^2\right]} = K_2 \frac{(s + a)}{\left[(s + a)^2 + b^2\right]} + \frac{(K_3 - K_2 a)}{b} \frac{b}{\left[(s + a)^2 + b^2\right]}$$

And $$L^{-1}\left\{\frac{K_2 s + K_3}{\left[(s + a)^2 + b^2\right]}\right\} = K_2 e^{-at}\cos(bt) + \frac{(K_3 - K_2 a)}{b} e^{-at}\sin(bt)$$