# OPTIMAL EXPLOITATION OF FLUCTUATING TARGET MEASUREMENTS

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## ABSTRACT

This paper describes an approach for evaluating the likelihood of received measurements with a fluctuating amplitude source. The likelihood drives higher level signal processing mechanisms like trackers. We investigate targets with fluctuating radar cross section (RCS) characterized by Swerling models and derive a general closed-form expression for the likelihood ratio. It does not appear that this closed form solution is well known, despite the widespread use of Swerling models in tracking applications. We illustrate the technique on a radar target tracking simulation using a target with fluctuating scan-to-scan radar cross section.

***Index Terms*****—** nonlinear filtering, target tracking, Bayesian estimation, Swerling.

## 1. INTRODUCTION

Swerling models [1] are commonly used to describe fluctuation in Radar Cross Section (RCS) of a target in radar applications [2-6]. The Swerling I and III models assume target amplitude is constant through a scan, but varies from scan to scan. In particular, Swerling I describes scan-to-scan variation using an exponential distribution, while the Swerling III models the variation as Chi-squared.

In both cases, the target RCS is governed by an average RCS, $\sigma_{av}$. From scan to scan, the target RCS $\sigma$ is modeled as being drawn from a distribution defined by this $\sigma_{av}$. The actual received measurement is then described as coming from a distribution defined by $\sigma$. The Swerling II and IV models assume the variation is pulse to pulse in analogous form. Finally, the so-called Swerling V (or 0) model assumes the target RCS is constant.

Of interest in this paper is computation of the likelihood ratio based on "raw" (non-thresholded) received measurements for a target exhibiting Swerling fluctuations. We assume the target RCS at a scan is unknown, but rather only the average RCS is known.

This likelihood ratio drives higher-level signal processing modules like a tracker. A tracker exploits RCS models through the likelihood, $L(z) = p_1(z)/p_0(z)$, where $z$ is the received target intensity and $p_1$ and $p_0$ are models of the received intensity when a target is present and absent, respectively. For radar applications, a common model of $p_1$ is through a Rician distribution [2, 4] where the target RCS enters through the parameters of the Rician. The standard model of $p_0$ is Rayleigh [3] with known background parameter, although sometimes other models are used [2].

Our approach computes the likelihood by marginalizing over the unknown parameter $\sigma$. This approach has been investigated before, e.g., [4] performs a brute-force sampling based marginalization of the density, while [5] attempts an analytical solution for a subset of the Swerling models. Others [6, 7] consider the Swerling I model and derive a closed form solution for the likelihood that follows from our generic expression.

This paper derives an analytical solution to the likelihood marginalization for the general Swerling model. We show via simulation how it performs in a tracker. Our simulation compares the performance of a track-before-detect method in the case where $\sigma$ is known exactly from scan-to-scan and one where only $\sigma_{av}$ is known and the likelihood is computed via marginalization.

## 2. RADAR STATISTICAL MODELS

We assume a pulsed radar aggregates a number of pulses over a coherent processing interval (CPI) and performs Fourier processing to generate a Range/Doppler Map (RDM) [8, 9]. Each pixel in the RDM corresponds to a potential Range and Doppler of a target. We model the intensity distribution in Range/Doppler pixels that do *not* contain a target (i.e., the background distribution) as Rayleigh with (known) parameter $\eta$, i.e.,

$$p_0(z) = \frac{z}{\eta} e^{-z^2/2\eta}, \quad z > 0. \tag{1}$$

We further model the intensity distribution in pixels that *do* contain a target (the target present distribution) as Rician with parameters $A$ and $\eta$. $A$ is a function of Radar and target parameters, including antenna patterns, gains, power, and target range. Of primary interest is the fact that $A$ is a function of the time-varying target reflectivity $\sigma$. We write $A(\sigma) = \sqrt{Cq(x)\sigma}$ where $C$ captures radar specific parameters and $q(x)$ captures target parameters. Then

$$p_1(z|\sigma) = \frac{z}{\eta} e^{-\frac{z^2+A(\sigma)^2}{2\eta}} I_0\left(\frac{zA(\sigma)}{\eta}\right), \qquad z > 0. \tag{2}$$

defines the pixel distribution when a target is present. Under the Swerling 0 model, $\sigma$ is constant. Under the other Swerling models, $\sigma$ follows a distribution defined by $\sigma_{av}$.

The generic Swerling model [10] is written

$$p(\sigma) = \frac{m^m}{\Gamma(m)\,\sigma_{av}^m} \sigma^{m-1} e^{-\frac{m\sigma}{\sigma_{av}}} \qquad \sigma > 0 \tag{3}$$

The models are defined for real $m > 0$. We focus on the special cases of Swerling I ($m = 1$) and III ($m = 2$). The limiting case as $m \to \infty$ corresponds to the constant $\sigma$ case, also called Swerling 0 or Swerling V.

Swerling I corresponds to modeling the target as having many nearly equivalent scatterers, and can be written more familiarly as the exponential density

$$p(\sigma) = \frac{1}{\sigma_{av}} e^{-\frac{\sigma}{\sigma_{av}}} \qquad \sigma > 0. \tag{4}$$

Likewise, Swerling III corresponds to modeling the target as one large scattering surface and several other small scattering surfaces. It has the Chi-Squared density

$$p(\sigma) = \frac{4\sigma}{\sigma_{av}^2} e^{-\frac{2\sigma}{\sigma_{av}}} \qquad \sigma > 0. \tag{5}$$

## 3. TRACKER LIKELIHOOD

With that as background, the target likelihood ratio in a pixel is the ratio of the target-present likelihood to the target-absent likelihood

$$L(z|\sigma) = \frac{p_1(z|\sigma)}{p_0(z)} = e^{-\frac{A(\sigma)^2}{2\eta}} I_0\left(\frac{zA(\sigma)}{\eta}\right), \tag{6}$$

and depends on the true scan RCS $\sigma$.

When σ is known, the likelihood is

$$L(z) = e^{-\frac{A(\sigma)^2}{2\eta}} I_0\left(\frac{zA(\sigma)}{\eta}\right). \tag{7}$$

Here we assume we do not know $\sigma$ but do know $\sigma_{av}$ which defines the density from which $\sigma$ is drawn. The unconditional likelihood is computed by marginalizing over $\sigma$. The derivation is given in the Appendix.

For the special case of Swerling I, the unconditional likelihood [6, 7] can be expressed

$$L_{S\,I}(z) = \frac{1}{\alpha\sigma_{av}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z^2 Cq(x)}{4\eta^2\alpha}\right)^k = \frac{1}{\alpha\sigma_{av}} e^{\frac{z^2 Cq(x)}{4\eta^2\alpha}}, \tag{8}$$

where we have defined

$$\alpha = \left(\frac{Cq(x)}{2\eta} + \frac{1}{\sigma_{av}}\right). \tag{9}$$

In an analogous manner, for Swerling III we have

$$L_{S\,III}(z) = \frac{4}{\sigma_{av}^2\beta^2} \mathrm{e}^{\frac{z^2 Cq(x)}{4\eta^2\beta}} \left(\frac{z^2 Cq(x)}{4\eta^2\beta} + 1\right) \tag{10}$$

where

$$\beta = \left(\frac{Cq(x)}{2\eta} + \frac{2}{\sigma_{av}}\right). \tag{11}$$

## 4. SIMULATION

We illustrate the approach by tracking a target fluctuating in a manner described by the Swerling III model with a track-before-detect particle filter [9]. The target trajectory is a simulated maneuvering aircraft. The simulation synthesizes measurements from a pulsed Doppler radar operating at X-band using a $3ms$ CPI. The chosen parameters lead to $30m$ range resolution and $7m/s$ range-rate resolution with ambiguities at $7500m$ and $200m/s$ [8]. The RDM is then a collection of $250 \times 60$ pixels, one of which contains the target (under the so-called "small target approximation") and the rest which contain only background noise.

We simulate measurements according to the Swerling III model as follows. At each scan, the RCS parameter $\sigma$ is chosen from the distribution defined by $\sigma_{av}$ via eq. (5). Next, the received measurement amplitude is drawn using $\sigma$ via eq. (2). The following figures compare four approaches to computing the likelihood ratio in the tracker. For all cases in the simulation, the true $\sigma_{av} = 8$.

In the first case (the "fully informed likelihood" tracker), the tracker knows the true value of $\sigma$ at each scan and computes the likelihood comes from eq. (7). This provides the best tracking performance as it has access to the most information.

In the second case (the “naïve likelihood”), we assume we know $\sigma_{av}$ and simply use that in the scan-to-scan likelihood of eq. (7). This represents a mismatch from the measurement generation and the Bayes update and as such performs more poorly.

In the third case (the "marginalized likelihood" tracker), we again assume the tracker knows the average RCS $\sigma_{av}$ and instead compute the likelihood via marginalization from eq. (10) and eq. (11).

Finally, we illustrate a fourth case (the "presumed RCS likelihood" tracker), which does not know $\sigma_{av}$ precisely but uses a (potentially incorrect) value to perform the marginalization. Again, the likelihood comes from eq. (10) and eq. (11), but this time the filter presumption of $\sigma_{av}$ is mismatched to the true $\sigma_{av}$.

In all cases, the performance of the target tracker is measured by Root Mean Squared Error (RMSE). The trackers are initialized with the true target state plus some uncertainty. In the current simulation, which uses a standard sampling importance resampling filter (SIR), the computational cost is proportional to the number of particles in the filter.

Figure 1 compares the fully informed case where the true scan-to-scan $\sigma$ is known to the naïve approach where we simply use $\sigma_{av}$ in the scan-to-scan likelihood of eq. (7). We find the naïve approach requires substantially more samples to reach the same performance as the fully informed likelihood.

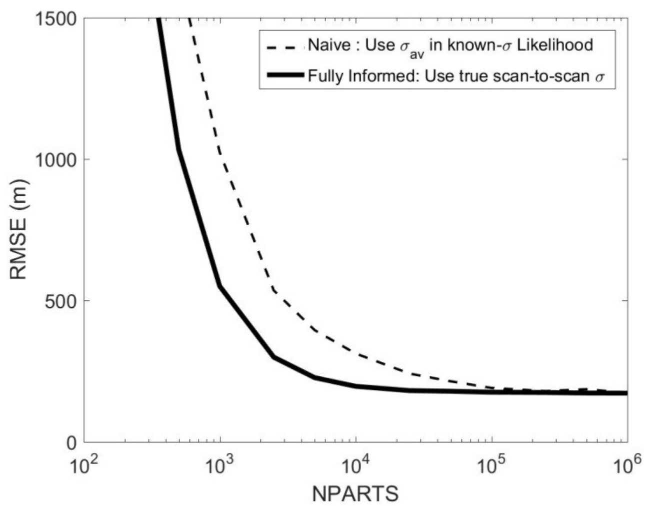


Fig 1. The fully informed tracker compared to the naïve tracker with $\boldsymbol{\sigma_{av} = 8}$. The naïve tracker requires significantly more particles (computational resources) to match the performance of the fully informed tracker.

Figure 2 compares the fully informed approach where the scan-to-scan $\sigma$ is known with the marginalized approach where only $\sigma_{av}$ is known and the marginalized likelihood is applied. We find that both approaches reach the same performance. The marginalization based approach requires just a small fraction of additional particles due to the unknown scan-to-scan RCS.

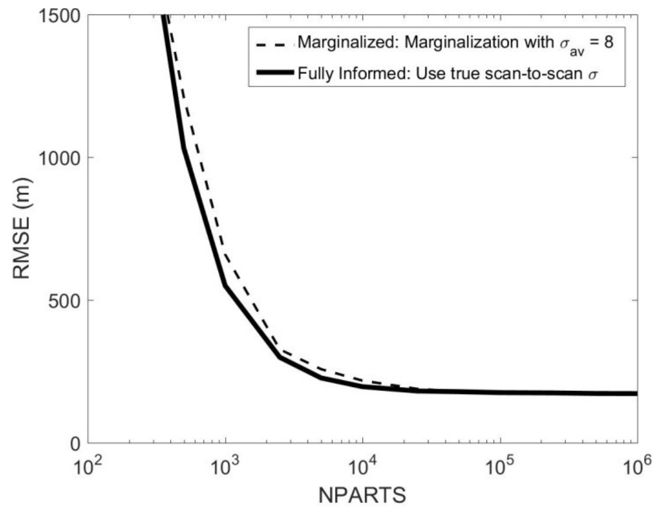


Fig 2. The fully informed vs. marginalized tracker, with $\boldsymbol{\sigma_{av} = 8}$. Tracking with unknown scan-to-scan $\boldsymbol{\sigma}$ matches the performance with known $\boldsymbol{\sigma}$ when marginalizing with a known $\boldsymbol{\sigma_{av}}$.

Finally, Figure 3 shows the performance of the marginalized tracker when the true value of $\sigma_{av}$ is not known precisely. We also show the (ideal) performance for the known scan-to-scan $\sigma$ case for reference. We find that the tracker performs well even when using a value for $\sigma_{av}$ that is incorrect.

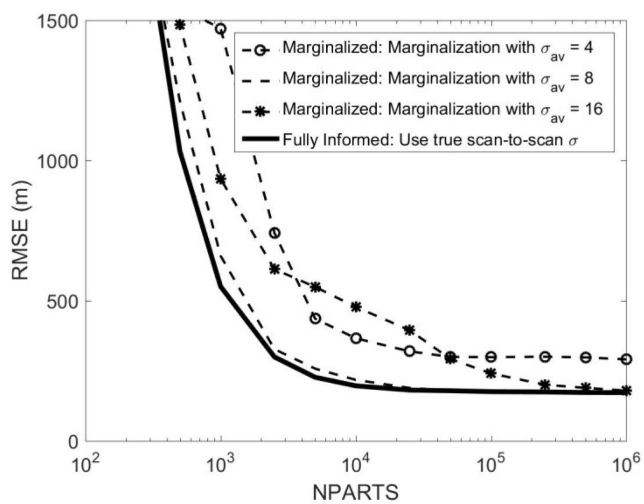


Fig 3. The fully informed vs. presumed $\boldsymbol{\sigma_{av}}$ marginalized tracker. Using the incorrect value of $\boldsymbol{\sigma_{av}}$ in the marginalization still produces a viable tracker when the estimate is near the true value.

## 4. CONCLUSION

This paper has described an approach for evaluating the likelihood of received measurements following the Swerling distribution. We derive a closed-form expression for the likelihood ratio in the Rayleigh/Rician scenario. We have illustrated the solution with a radar target tracking problem with a target with fluctuating scan-to-scan radar cross section.

## APPENDIX

This appendix derives the general unconditional likelihood by marginalizing over the unknown $\sigma$, which is specified by the Swerling model of eq. (3). Specifically, we seek to find a closed-form solution of the following integral:

$$L(z) = \int L(z|\sigma)p(\sigma)d\sigma, \tag{A.1}$$

where

$$L(z|\sigma) = \frac{p_1(z)}{p_0(z)} = e^{-\frac{A(\sigma)^2}{2\eta}} I_0(zA(\sigma)/\eta) \tag{A.2}$$

and

$$p(\sigma) = \frac{m^m}{\Gamma(m)\,\sigma_{av}^m}\sigma^{m-1}e^{-\frac{m\sigma}{\sigma_{av}}} \qquad \sigma > 0. \tag{A.3}$$

Using eq. (A.1) and the definition $A(\sigma) = \sqrt{Cq(x)\sigma}$ we can proceed directly as:

$$\begin{aligned} L(z) &= \int L(z|\sigma)p(\sigma)d\sigma \\ &= \int_0^\infty d\sigma e^{-\frac{A(\sigma)^2}{2\eta}} I_0\left(\frac{zA(\sigma)}{\eta}\right)\frac{m^m}{\Gamma(m)\,\sigma_{av}^m}\sigma^{m-1}e^{-\frac{m\sigma}{\sigma_{av}}} \\ &= \frac{m^m}{\Gamma(m)\,\sigma_{av}^m}\int_0^\infty d\sigma e^{-\gamma\sigma}\sigma^{m-1}\sum_{k=0}^\infty \frac{1}{k!\,k!}\left(\frac{\delta\sigma}{4}\right)^k \\ &= \frac{m^m}{\Gamma(m)\,\sigma_{av}^m}\sum_{k=0}^\infty \frac{1}{k!\,k!}\left(\frac{\delta}{4}\right)^k \int_0^\infty d\sigma e^{-\gamma\sigma}\sigma^{m+k-1} \\ &= \frac{m^m}{\Gamma(m)\,\sigma_{av}^m\gamma^m}\sum_{k=0}^\infty \frac{\Gamma(m+k)}{k!\,k!}\left(\frac{\delta}{4\gamma}\right)^k \\ &= \frac{m^m}{\Gamma(m)\,\sigma_{av}^m\gamma^m}\sum_{k=0}^\infty \frac{\Gamma(m+k)\Gamma(m)}{\Gamma(k+1)\Gamma(m)}\frac{\left(\frac{\delta}{4\gamma}\right)^k}{k!} \\ &= \frac{m^m}{\sigma_{av}^m\gamma^m}\,{}_1\tilde{F}_1\left(m; 1; \frac{\delta}{4\gamma}\right). \end{aligned} \tag{A.4}$$

Where we have defined

$$\gamma(m) = \frac{Cq(x)}{2\eta} + \frac{m}{\sigma_{av}} \quad \text{and} \quad \delta = \frac{z^2Cq(x)}{\eta^2}. \tag{A.5}$$

The derivation exploits an infinite series expansion of the modified Bessel function (A.6), the integral definition of the gamma function (A.7), and a regularized confluent hypergeometric function (A.8).

$$I_0(x) = \sum_{k=0}^\infty \frac{1}{k!\,k!}\left(\frac{x}{2}\right)^{2k}, \tag{A.6}$$

$$\Gamma(t) = \int_0^\infty dx\, x^{t-1}e^{-x}, \tag{A.7}$$

$${}_1\tilde{F}_1(a; b; x) = \sum_{k=0}^\infty \frac{\Gamma(a+k)}{\Gamma(b+k)\Gamma(a)}\,\frac{z^k}{k!}. \tag{A.8}$$

In the case that $m$ is a positive integer, the final expression in eq. (A.4) has a succinct form which is a finite polynomial multiplied by an exponential:

$$\begin{aligned} L(z) &= \frac{m^m}{\sigma_{av}^m\gamma^m}\mathrm{L}_{-\mathrm{m}}\left(\frac{\delta}{4\gamma}\right) \\ &= \frac{m^m}{\sigma_{av}^m\gamma^m}\mathrm{e}^{\frac{\delta}{4\gamma}}\sum_{l=0}^{m-1}\binom{m-1}{l}\left(\frac{\delta}{4\gamma}\right)^l\frac{1}{l!}, \end{aligned} \tag{A.9}$$

which follows from the mathematical relationships of associated Laguerre polynomials (A.10) and the negative index abstraction of the Laguerre polynomials (A.11):

$$L_n^m(x) = \frac{(m+n)!}{m!\,n!}\,{}_1\tilde{F}_1(-n; m+1; x), \tag{A.10}$$

$$L_{-n}(x) = e^x\, L_{n-1}(-x). \tag{A.11}$$

The special cases of Swerling I and III can be expressed as given in the main article by setting $m = 1$, and $2$ in (A.9) respectively.

## 5. REFERENCES


[1] Swerling, P. Probability of Detection for Fluctuating Targets. IRE Transactions on Information Theory, vol.6, no. 2, 1960.

[2] M. Richards, Fundamentals of Radar Signal Processing, Second Edition (McGraw-Hill Professional Engineering), 2014.

[3] Skolnik, M. Introduction to Radar Systems: Third Edition. McGraw-Hill, New York, 2001.

[4] Rutten, M. G., Gordon, N.J. and Maskell, S. "Recursive track-before-detect with target amplitude fluctuations", *IEE Proc.-Radar Sonar Navigation*, Vol. 152, No. 5, October 2005.

[5] M. Mcdonald and B. Balaji, "Track-before-Detect Using Swerling 0, 1, and 3 Target Models for Small Manoeuvring Maritime Targets", *EURASIP Journal on Advances in Signal Processing*, 2008.

[6] Lepoutre, A., Rabaste O., and Le Gland, F., "Exploiting Amplitude Spatial Coherence for Multi-target Particle Filter in Track-Before-Detect", *IEEE Fusion Conference,* 2013.

[7] Y. Boers and J. N. Driessen, "Multitarget particle filter track before detect application", *IEE Proceedings - Radar, Sonar, and Navigation*, vol. 151, no. 6, 2004.

[8] C. Kreucher, K. Bell, D. Sobota, "A Comparison of Tracking Algorithms for Supermaneuverable Targets", *Proceedings of the 2015 IEEE Fusion Conference*, July 6-9 2015.

[9] C. Kreucher, "Exploiting Narrowband Bistatic Radar Measurements for Dismount Detection and Tracking". *IEEE Antennas and Propagation Magazine*, 53(1):98-105, Feb. 2011.

[10] P. Swerling, "Radar Probability of Detection for Some Additional Fluctuating Target Cases", *IEEE Trans. Aerospace and Electronic Systems*, vol. 33, no. 2, pp. 698-709, April 1997.