CRB-Optimal Sensor Placement for Multiple Passive Acoustic Arrays

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***Abstract*—This paper describes a method for selecting the location of multiple passive acoustic arrays to optimize information collection. The passive acoustic arrays are used to localize a contact, and our goal is to choose how to situate and orient the arrays to optimize this localization estimate. To this end, we derive the Cramér-Rao bound on the variance of the estimate of a target's location as a function of where the sensing resources are placed. We then use this bound to select where to place the sensors. We illustrate the method by selecting where to place and orient 5 arrays in a region.**

***Index Terms*—Efficient Information Collection, Sensor Placement Optimization, Cramér-Rao bound, Passive Arrays.**

I. Introduction

This paper describes an approach for choosing where to place passive acoustic arrays to optimize information collection. The problem of managing sensing resources for optimal information gathering has recently received increased interest in the literature [1], [2]. As [2] discusses, many popular approaches to sensor resource allocation rely on information theoretic measures. Some important work includes methods based on Fisher information [3], the Kullback-Leibler [4] divergence and the Rényi entropy [5]. There are a wide variety of applications of sensor management techniques, including waveform selection, motion planning, and sensor placement [1]. The techniques have been applied using acoustic, radar, and optical sensors [2]. Particularly interesting applications include those with heterogeonous sensors or multiple modalities which measure different types of information (e.g., classification information and location information) where the usage must be traded against each other [8], [9].

This paper focuses on a domain where multiple arrays are used to localize an acoustic contact. Our goal is to choose where to situate and orient the arrays to optimize this localization estimate. Our approach is to compute the Cramér-Rao bound (CRB) on the variance of the estimate of a target's location as a function of where the sensing resources are placed. We then use this bound to select where to place the sensors.

This paper proceeds as follows. First, we develop a statisical model describing how sensor measurements couple to the target location. Then we derive the CRB bound on estimate variance as a function of where the arrays are placed. Finally, we illustrate with a simulation how to select the location of the arrays to optimize localization performance.

II. Passive Acoustic Array Sensing Model

Assume an N element acoustic array is centered at (x_c, y_c) with orientation ϕ and distance d between the individual elements. The element positions can be written, for $n = 0 \cdots N-1$, as

$$x[n] = x_c + \left(\frac{2n-(N-1)}{2}\right) d\cos(\phi) \tag{1}$$
$$y[n] = y_c + \left(\frac{2n-(N-1)}{2}\right) d\sin(\phi).$$

Assume further that a single acoustic source is positioned at (x, y) and emits at a single frequency f. The energy emitted from the source arrives at the sensors at different times due to the physical distance. Let $r[n]$ denote the range from the n^{th} sensor (hydrophone) to the source, i.e.,

$$r[n] = \sqrt{\big(x - x[n]\big)^2 + \big(y - y[n]\big)^2}. \tag{2}$$

The $N \times 1$ vector r will denote the collection of ranges $r[n]$. The signal that impinges on hyrdrophones will be denoted by s which is an $N \times 1$ vector having elements

$$s[n] = e^{-jkr[n]}, \tag{3}$$

where $k = 2\pi f/c$. Both s and r depend on the source location (x, y). This will be supressed for readability. The model assumes a unit amplitude signal and zero initial phase. We comment on the impact of these assumptions later.

This nominal model does not include attenuation due to spreading. A spherical spreading model [7], leads to

$$s_{spherical}[n] = \frac{\alpha}{r[n]^2} e^{-jkr[n]}, \tag{4}$$

and a cylindrical spreading model gives

$$s_{cylindrical}[n] = \frac{\alpha}{r[n]} e^{-jkr[n]}, \tag{5}$$

where α corresponds to the reference intensity.

We have chosen to ignore absorption, which means the model is strictly applicable only at low frequencies (e.g., $< 5kHz$). The model can be generalized to include this factor and others and the derivation which follows is only slightly modified.

Each sensor makes measurements of the incident energy corrupted by additive noise. Let the noise corrupted measurement at sensor n be called $z[n]$, where

$$z[n] = s[n] + w[n]. \tag{6}$$

Our statisical model is that $w[n]$ is a circular symmetric complex normal, i.e., $w[n] \sim CN(0, \sigma^2)$. The collection of noise realizations and measurements will be denoted using the $N \times 1$ vectors w and z in analogy with s and r.

With this as background, the model for the received data can be compactly stated as

$$p(z|x,y) = \frac{1}{\pi^N \sigma^{2N}} e^{-(z-s)^H(z-s)/\sigma^2}. \tag{7}$$

III. The Passive Acoustic CRB

Assume we have an unbiased estimator of x and y from the data z. The Cramér-Rao Bound on the variance of any such estimator is given by the inverse of the Fisher Information Matrix (FIM)

$$I = -\begin{bmatrix} \mathbb{E}\left[\frac{\partial^2}{\partial x^2}\ln p(z|x,y)\right] & \mathbb{E}\left[\frac{\partial^2}{\partial x \partial y}\ln p(z|x,y)\right] \\ \mathbb{E}\left[\frac{\partial^2}{\partial y \partial x}\ln p(z|x,y)\right] & \mathbb{E}\left[\frac{\partial^2}{\partial y^2}\ln p(z|x,y)\right] \end{bmatrix}. \tag{8}$$

Our derivation will follow the standard approach [6], which was applied in the past to derive bounds in the single array far-field angle estimation only case.

From eq. (7), we can write

$$\ln p(z|x,y) = C + \frac{z^H s + s^H z - z^H z - s^H s}{\sigma^2} \tag{9}$$

and

$$\frac{\partial}{\partial x}\ln p(z|x,y) = \frac{z^H s_x + s_x^H z - s^H s_x - s_x^H s}{\sigma^2}, \tag{10}$$

where s_x is shorthand for $\frac{\partial}{\partial x}s$. The y partial is analogous. From here forward, we omit explicit definition of the y terms because they are obvious by inspection. Continuing,

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\ln p(z|x,y) &= \\ &\frac{z^H s_{xx} + s_{xx}^H z - s^H s_{xx} - 2 s_x^H s_x - s_{xx}^H s}{\sigma^2} \\ \frac{\partial^2}{\partial x \partial y}\ln p(z|x,y) &= \\ &\frac{z^H s_{xy} + s_{xy}^H z - s^H s_{xy} - s_y^H s_x - s_{xy}^H s - s_x^H s_y}{\sigma^2}. \end{aligned} \tag{11}$$

s_{xx} and s_{xy} are shorthand for $\frac{\partial^2}{\partial x^2}s$ and $\frac{\partial^2}{\partial x \partial y}s$, respectively. The expectations with respect to the random variable Z are

$$\begin{aligned} \mathbb{E}\left[\frac{\partial^2}{\partial x^2}\ln p(z|x,y)\right] &= -2 s_x^H s_x/\sigma^2 \\ \mathbb{E}\left[\frac{\partial^2}{\partial x \partial y}\ln p(z|x,y)\right] &= -(s_y^H s_x + s_x^H s_y)/\sigma^2 \end{aligned} \tag{12}$$

We can write the expectations explicitly as a function of the elements of s as

$$\mathbb{E}\left[\frac{\partial^2}{\partial x^2}\ln p(z|x,y)\right] = \frac{-2}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial}{\partial x}s[n]^H \frac{\partial}{\partial x}s[n]. \tag{13}$$

and

$$\begin{aligned} \mathbb{E}\left[\frac{\partial^2}{\partial x \partial y}\ln p(z|x,y)\right] = \frac{-1}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial}{\partial y}s[n]^H \frac{\partial}{\partial x}s[n] + \\ \frac{\partial}{\partial x}s[n]^H \frac{\partial}{\partial y}s[n]. \end{aligned} \tag{14}$$

The partials of s come immediately from eqs. (3, 4, and 5). It is clear at this point that the assumption on initial signal phase in eqs. (3, 4, 5) has no bearing on the CRB because each term in the FIM involves only products of $s[n]$ and $s[n]^H$. The assumption of a unit amplitude signal does have impact. If the signal had amplitude a, each expectation has an additional factor of a^2. One way to view this is that the (known) amplitude is subsumed into the (known) σ^2 which is now to be viewed as a noise-to-signal power ratio.

For clarity, we first specialize to the no-spreading case and later show the generalization to spherical and cylindrical spreading. For the no-spreading case we have

$$\frac{\partial}{\partial \cdot}s[n] = e^{-jkr[n]}(-jk)\frac{\partial}{\partial \cdot}r[n]. \tag{15}$$

The products needed to evaluate the expectation of eqs. (13) and (14) are

$$\frac{\partial}{\partial x}s[n]^H \frac{\partial}{\partial x}s[n] = k^2\left(\frac{\partial}{\partial x}r[n]\right)^2 \tag{16}$$

and

$$\frac{\partial}{\partial x}s[n]^H \frac{\partial}{\partial y}s[n] = k^2 \frac{\partial}{\partial x}r[n]\frac{\partial}{\partial y}r[n]. \tag{17}$$

We can then write the elements of I (see eq. (8)) in terms of the partials of r in the no-spreading case as

$$I = \frac{2k^2}{\sigma^2}\begin{bmatrix} \sum\limits_{n=0}^{N-1}\left(\frac{\partial}{\partial x}r[n]\right)^2 & \sum\limits_{n=0}^{N-1}\frac{\partial}{\partial x}r[n]\frac{\partial}{\partial y}r[n] \\ \sum\limits_{n=0}^{N-1}\frac{\partial}{\partial x}r[n]\frac{\partial}{\partial y}r[n] & \sum\limits_{n=0}^{N-1}\left(\frac{\partial}{\partial y}r[n]\right)^2 \end{bmatrix}. \tag{18}$$

When spreading loss is considered, the FIM elements generalize to the forms

$$I^{xx}_{spherical} = \frac{2\alpha^2}{\sigma^2}\sum_{n=0}^{N-1}\frac{k^2 r[n]^2 + 4}{r[n]^6}\left(\frac{\partial}{\partial x}r[n]\right)^2 \tag{19}$$

and

$$I^{xx}_{cylindrical} = \frac{2\alpha^2}{\sigma^2}\sum_{n=0}^{N-1}\frac{k^2 r[n]^2 + 1}{r[n]^4}\left(\frac{\partial}{\partial x}r[n]\right)^2, \tag{20}$$

where I^{xx} refers to the upper left element of I. The other elements of the FIM are analogous.

The FIM is completely specified by noting from eq. (2)

$$\frac{\partial}{\partial x} r[n] = \frac{x - x[n]}{r[n]} \tag{21}$$

and

$$\frac{\partial}{\partial y} r[n] = \frac{y - y[n]}{r[n]}, \tag{22}$$

giving, for the no-spreading case,

$$I = \frac{2k^2}{\sigma^2} \begin{bmatrix} \sum_{n=0}^{N-1} \frac{(x-x[n])^2}{r[n]^2} & \sum_{n=0}^{N-1} \frac{(x-x[n])(y-y[n])}{r[n]^2} \\ \sum_{n=0}^{N-1} \frac{(x-x[n])(y-y[n])}{r[n]^2} & \sum_{n=0}^{N-1} \frac{(y-y[n])^2}{r[n]^2} \end{bmatrix}. \tag{23}$$

A. The Single Array Bound

For compactness, we continue with the no-spreading case. The spreading loss models are analogous. The single-sensor CRB on position localization is given by the inverse of the FIM,

$$C \geq C_0 \begin{bmatrix} \sum_{n=0}^{N-1} \frac{(y-y[n])^2}{r[n]^2} & -\sum_{n=0}^{N-1} \frac{(x-x[n])(y-y[n])}{r[n]^2} \\ -\sum_{n=0}^{N-1} \frac{(x-x[n])(y-y[n])}{r[n]^2} & \sum_{n=0}^{N-1} \frac{(x-x[n])^2}{r[n]^2} \end{bmatrix} \tag{24}$$

where

$$C_0 = \frac{\sigma^2}{2k^2} \Bigg(\sum_{n=0}^{N-1} \frac{(x-x[n])^2}{r[n]^2} \sum_{n=0}^{N-1} \frac{(y-y[n])^2}{r[n]^2} - \Big(\sum_{n=0}^{N-1} \frac{(x-x[n])(y-y[n])}{r[n]^2} \Big)^2 \Bigg). \tag{25}$$

This represents the localization performance possible from exploiting wavefront curvature. Qualitatively, it improves (is reduced) with any of the following : (i) the noise (σ^2) is reduced; (ii) the signal frequency (k) is increased; (iii) the range to the contact (r) is decreased; (iv) the inter-element spacing (d) is increased. Furthermore, the bound reflects the fact that some geometrical arrangements do not allow bounded-variance unbiased estimation. For example, a linear array will not be able to estimate the range of a target at endfire. This is seen by the singularity of the FIM in this case. For example, if the array has $x[n] = 0$ for all n and a contact is located at $x = 0$, the determinant C_0 is 0 because each term $x - x[n]$ becomes 0.

The trace of C is the best mean squared error (MSE) of any unbiased estimator. As an example, Figure 1 shows the MSE for contacts located over a $20km \times 20km$ region when a single 101 element array is placed at the origin with inter-element spacing $\lambda/2$. The target has unit amplitude and we selected $\sigma = 1$.

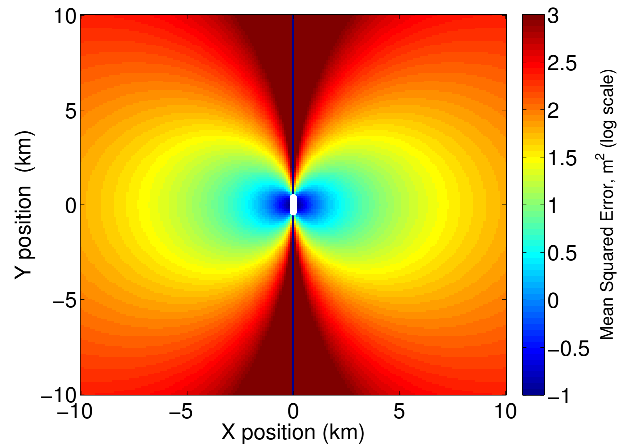

Fig. 1. The MSE Bound of an unbiased estimator for the single-sensor no-spreading case. The array is shown in white and is oriented vertically at the origin. Note that contacts located directly above or below are unobservable. This is indiacted by the blue vertical stripe at $x = 0$.

B. The Multiple Array Bound

For two acoustic arrays with measurement vectors z_1 and z_2 (independent conditioned on x and y), we can write

$$p(z_1, z_2|x, y) = p(z_1|x, y)p(z_2|x, y). \tag{26}$$

Which leads to the FIM $I = I_1 + I_2$. With M arrays, we have $I = \sum_{m=0}^{M-1} I_m$. The corresponding CRB can be computed by inverting this sum, giving a form of the bound analogous to that of eq. (24).

As an example, Figure 2 shows the MSE for targets located over a $20km \times 20km$ region when two acoustic arrays are present. One array is centered at the origin and the other at $x = -5km$. The inter-element spacing is $\lambda/2$. One array is oriented at $0°$ and the other at $90°$. The target has unit amplitude and $\sigma = 1$ again.

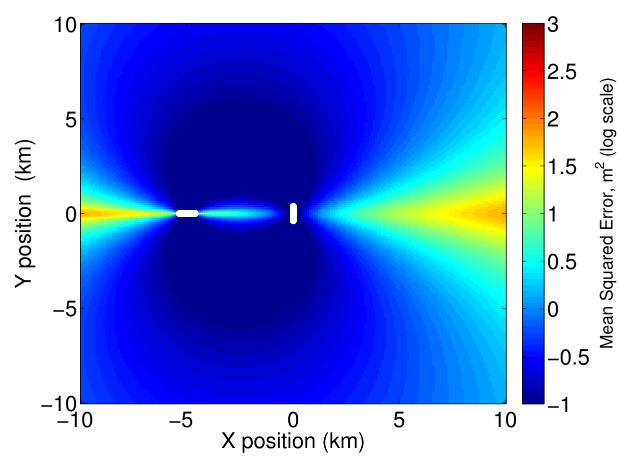

Fig. 2. The MSE Bound For the Two-Sensor Case. One array is oriented vertically at the origin and the other horizontally at -5km.

C. CRB-driven Sensor Placement

We use the multisensor CRB to determine where to place multiple passive acoustic arrays. The method proceeds as follows. For M arrays, a potential array placement is defined by the $3M$ vector $[x_1, y_1, \phi_1, \cdots, x_M, y_M, \phi_M]$. For this $3M$ vector there is a MSE map over the region of interest like those shown in Figures 1 and 2. This map describes the estimation MSE if the target was actually located at each of those points. We characterize this map by the worst-case bound over the entire region. As such, each placement of the M arrays corresponds to a real-valued number which characterizes its utility.

We select the location of the M sensors so as to minimize this worst case error bound. This optimization requires a (bounded) search in $3M$-dimensional space which has no convexity guarantees. In this study, we use a meta-heuristic algorithm called Cuckoo Search [10] to perform the optimization.

Figure 3 shows the optimal placement of 5 arrays in a $20km \times 20km$ region when the no-spreading model is used. The placement of the arrays was constrained to be inside the region. It is intuitive that the best bound is achieved when the sensors are placed along the edges of the region and as far from each other as possible.

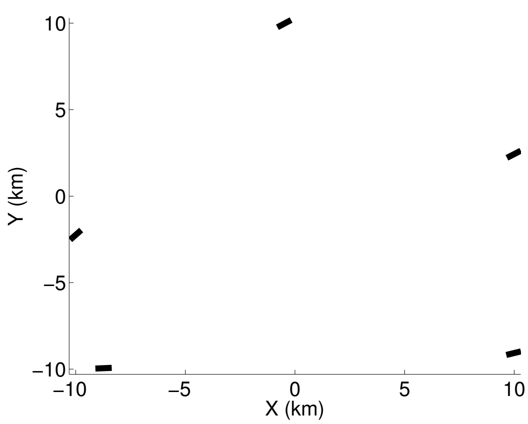

Fig. 3. Best placement of 5 arrays, under the no-spreading model.

Figures 4 and 5 show the selected placements under the cylindrical and spherical spreading models, respectively. In these cases, the best bound is achieved when the sensors are moved inward to prevent any position from having an excessively long range. This is balanced against the desire to spread the sensors as far apart for maximum angluar diversity.

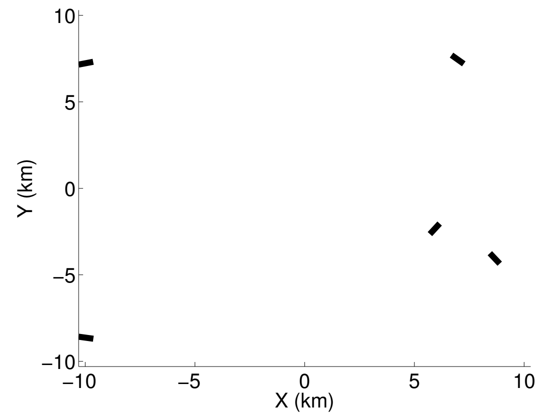

Fig. 4. Best placement of 5 arrays, under the cylindrical spreading model.

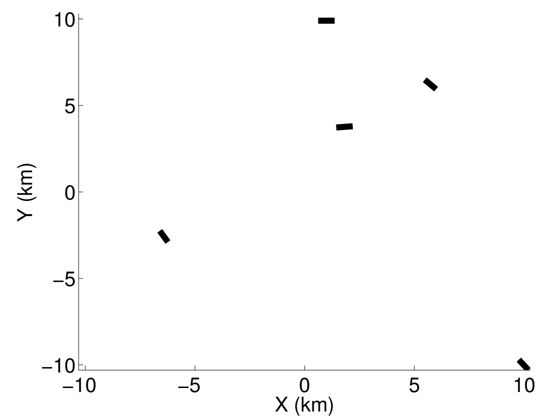

Fig. 5. Best placement of 5 arrays, under the spherical spreading model.

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