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PDF target detection and tracking

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ABSTRACT

This paper presents a new approach to the real time single and multiple target detection and tracking problems with measurement input data. The new approach addresses the measurement uncertainty-of-origin issue by capturing all measurement input data information in the Bayesian conditional probability density function (PDF), used in the recursive propagation of the posterior target detection and tracking information PDF over time via Bayesian and Markov PDF updates. The application of Bayes' formula over time resolves measurement association ambiguities. Under linear, Gaussian assumptions, the posterior PDF is a repeating Gaussian mixture. This leads to computational simplifications and efficiencies in implementation. Simulation results demonstrate operation and performance.

1. Introduction and overview

The primary purpose of many systems is detection, localization, and classification of targets of interest. This paper is concerned with a real time solution for the first two: detection and localization, or tracking a time-varying target parameter state such as position and velocity. To achieve these purposes, data from sensors, usually energy either reflected or emitted by a target, is collected and exploited.

The standard architecture for a detection and tracking system may be considered to have three high level functional blocks: (1) sensors (2) signal processing (3) detection and tracking. The first function is the collection of sensor data over time. The signal processing function accepts sensor data as input. It interrogates the computed sensor data observation space for the presence of potential targets and produces "measurements" at potential target detection locations over time as output. This process may be thresholding of energy peaks.

Measurements are the input data to the last function, system level detection and tracking, which provides the system outputs, the number of detected targets and the target parameter state of each detected target, over time. This paper focuses on the detection and tracking function and presents a new approach.

A major issue is measurement "uncertainty-of-origin". A measurement may either be due to ("associated" with) a target of interest but with a random observation error or may be unwanted random "clutter" (false alarm). Even for the former desired case, there is uncertainty-of-origin with respect to multiple targets of interest. Additionally, targets may be "missed", or not detected in the measurement input data.

A target parameter state cannot be deterministically found due to this uncertainty and randomness in the measurement input data. Even if a target parameter state were known with certainty at a time, a future target parameter state cannot be found with certainty due to random target time-varying behavior. These statements assume target presence and can also be applied to the target detection problem itself, the determination of a target detection state: target absent or target present.

The standard accepted approach to address the ill effects of uncertainty and randomness for this or any problem is to statistically characterize the input data and

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also for this problem, the time-varying, or dynamic, target behavior. Then, the desired target information (current target detection and parameter states) is statistically characterized using applied statistical inference. I.e., the probability density function (PDF) of the desired target information conditioned on knowing all available input data (current and past) is found and is the posterior target information PDF, which contains all target knowledge. This is our basic approach.

The standard accepted approach to achieve this is the recursive propagation of the posterior target information PDF over time via Bayesian and Markov PDF updates. The completely general Bayes–Markov filter is presented in [1] and is beyond any question as the correct and preferred approach. The development is the incorporation of the input data (measurements here) observation model into the Bayesian conditional PDF and the target dynamic model into the Markov transition PDF. They are required in the Bayesian and Markov PDF updates. Then, algebra can be performed in the general Bayes–Markov filter to proceed to a solution.

The Kalman filter [2,3] is the prime example for this fundamental approach and is a special case of the more general and complex problem here. The goal is to repeat this success. The strict adherence to the general Bayes–Markov filter is what sets this paper apart from the major existing approaches, probabilistic data association (PDA) [4], joint probabilistic data association (JPDA) [5], and multi-hypotheses tracking (MHT) [6], which explicitly enumerate association hypotheses. This paper is related to our previous work [7–16]. Comprehensive treatments of this field are [17–23]. Specific applications include [24–26].

This paper presents solutions for both the single and multiple target detection and tracking problems. The single target solution is presented in two levels of specificity. The first level is the general single target detection and tracking solution, presented in Section 2, as a special case of the general Bayes–Markov filter. This solution is realized as simultaneous interdependent detection and tracking solutions and was developed in [9]. The final level of specificity, presented in Section 3, is the incorporation of the standard uncertainty-of-origin measurement observation model and the standard target dynamic model both with linear, Gaussian assumptions (i.e., the Kalman filter conditions) [17–23] into the general solution in Section 2. This solution is optimal in that it identically achieves the correct posterior PDF solution, which contains all detection and tracking information, and is necessary and sufficient to optimally compute any desired detection and tracking output.

The straightforward joint extension to the multiple target problem is, in general, not computationally feasible. A single target solution depends only on its correctly associated measurements. They are identified as part of the solution and can be efficiently excluded from all other single target solutions. The resulting single target solutions are all mutually orthogonal. They do not “share” measurements. This is also the underlying basis for all the major existing approaches. These single target solutions are independent with each solution optimal. Then, the optimal joint multiple target posterior PDF solution is the product over all these single target solutions. The multiple target solution, presented in Section 4, consists of interactive parallel single target solutions, or detector-trackers, with each tracking its “ own” target. The basis of this approach was developed in [10].

2. General single target detection and tracking solution

In this section, we develop the optimal general single target detection and tracking solution. We cast the problem of estimating target presence and its parameter state conditioned on its presence as one of probabilistic inference. Using statistical models on input data and target kinematics, we show how these probabilities can be recursively computed from the actual received data.

2.1. Setup

Input data are received over time at discrete processing time intervals, or scans. Let $\mathbf{z}_k$ be all input data at scan k occurring at time t_k and is general. Let $\mathbf{Z}_k$ be all input data through scan k and is the union of all current and past input data $\mathbf{z}_k$.

$$\mathbf{Z}_k = \mathbf{z}_k \cup \mathbf{z}_{k-1} \cup \mathbf{z}_{k-2} \cup \cdots = \mathbf{z}_k \cup \mathbf{Z}_{k-1} \tag{1}$$

Let $\mathfrak{X}_k$ be the time varying total state of all target information at scan k. For the single target detection and tracking problem, the observable total target state $\mathfrak{X}_k$ is the joint state of discrete target detection state h_k and general continuous target parameter state $\mathbf{x}_k$. Target detection state h_k has two possible values: target absent and present hypotheses $\mathrm{H}_{0,k}$,$\mathrm{H}_{1,k}$. Target parameter state $\mathbf{x}_k$ is not observable for target absent, only for target present. Posterior total target state PDF $\mathrm{p}(\mathfrak{X}_k|\mathbf{Z}_k)$ is

$$\mathrm{p}(\mathfrak{X}_k|\mathbf{Z}_k)|_{h_k=\mathrm{H}_{0,k}} = \mathrm{p}(h_k,\mathbf{x}_k|\mathbf{Z}_k)|_{h_k=\mathrm{H}_{0,k}} = \mathrm{p}(\mathrm{H}_{0,k}|\mathbf{Z}_k) \tag{2}$$

$$\mathrm{p}(\mathfrak{X}_k|\mathbf{Z}_k)|_{h_k=\mathrm{H}_{1,k}} = \mathrm{p}(h_k,\mathbf{x}_k|\mathbf{Z}_k)|_{h_k=\mathrm{H}_{1,k}} = \mathrm{p}(\mathrm{H}_{1,k},\mathbf{x}_k|\mathbf{Z}_k) \tag{3}$$

Total target state conditional PDF $\mathrm{p}(\mathbf{z}_k|\mathfrak{X}_k)$ is

$$\mathrm{p}(\mathbf{z}_k|\mathfrak{X}_k)|_{h_k=\mathrm{H}_{0,k}} = \mathrm{p}(\mathbf{z}_k|h_k,\mathbf{x}_k)|_{h_k=\mathrm{H}_{0,k}} = \mathrm{p}(\mathbf{z}_k|\mathrm{H}_{0,k}) \tag{4}$$

$$\mathrm{p}(\mathbf{z}_k|\mathfrak{X}_k)|_{h_k=\mathrm{H}_{1,k}} = \mathrm{p}(\mathbf{z}_k|h_k,\mathbf{x}_k)|_{h_k=\mathrm{H}_{1,k}} = \mathrm{p}(\mathbf{z}_k|\mathrm{H}_{1,k},\mathbf{x}_k) \tag{5}$$

Posterior total target state PDF $\mathrm{p}(\mathfrak{X}_k|\mathbf{Z}_k)$ in (2) and (3) is a mixed discrete and continuous PDF. In (2), probability $\mathrm{p}(\mathrm{H}_{0,k}|\mathbf{Z}_k)$ is the posterior target absent probability. In (3), PDF $\mathrm{p}(\mathrm{H}_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ is the continuous joint posterior target present and target parameter state $\mathbf{x}_k$ PDF and is an improper PDF in that its total integral is one only for certain target presence. The joint posterior probability of target present and target parameter state $\mathbf{x}_k$ occurring in a region is the integral of posterior target parameter state PDF $\mathrm{p}(\mathrm{H}_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ in (3) over that region. Detection and tracking information in input data $\mathbf{z}_k$ is only available through (4) and (5), determined by observation models.

Posterior target present probability $\mathrm{p}(\mathrm{H}_{1,k}|\mathbf{Z}_k)$ irrespective of any target parameter state $\mathbf{x}_k$ value is the total probability in posterior target parameter state PDF $\mathrm{p}(\mathrm{H}_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ in (3).

$$\mathrm{p}(\mathrm{H}_{1,k}|\mathbf{Z}_k) = \int \mathrm{p}(\mathrm{H}_{1,k},\mathbf{x}_k|\mathbf{Z}_k)\,\mathrm{d}\mathbf{x}_k \tag{6}$$

Together, target absent and target present hypotheses $H_{0,k}$,$H_{1,k}$ have a two state discrete posterior detection PDF $p(H_{0,k}|\mathbf{Z}_k), p(H_{1,k}|\mathbf{Z}_k)$. From (2), (3), and (6),

$$\int p(\mathfrak{X}_k|\mathbf{Z}_k)\,d\mathfrak{X}_k = p(H_{0,k}|\mathbf{Z}_k) + \int p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)\,d\mathbf{x}_k = 1 \quad (7)$$

$$= p(H_{0,k}|\mathbf{Z}_k) + p(H_{1,k}|\mathbf{Z}_k) = 1 \quad (8)$$

The tracking only case is target parameter state $\mathbf{x}_k$ conditioned on target present hypothesis $H_{1,k}$ true. Continuous tracking only posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ is a normalized version (forced to integrate to 1) of posterior target parameter state PDF $p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ in (3).

$$p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k) = \frac{p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)}{\int p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)\,d\mathbf{x}_k} = \frac{p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)}{p(H_{1,k}|\mathbf{Z}_k)} \quad (9)$$

$$p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k) = p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)p(H_{1,k}|\mathbf{Z}_k) \quad (10)$$

In (9) and (10), posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ and posterior target parameter state PDF $p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ are related by posterior target present probability $p(H_{1,k}|\mathbf{Z}_k)$. Data $\mathbf{Z}_k$ as a conditional in this section generalizes to other conditionals.

2.2. Methodology overview

Posterior detection PDF $p(H_{0,k}|\mathbf{Z}_k), p(H_{1,k}|\mathbf{Z}_k)$ and posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$, respectively, contain all detection and tracking information. They are required for the system outputs in Section 1. Also, independent dynamic models are defined on target presence and the target parameter state conditioned on target present. For these reasons, both posterior detection PDF $p(H_{0,k}|\mathbf{Z}_k)$, $p(H_{1,k}|\mathbf{Z}_k)$ (see Section 2.3) and posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ (see Section 2.4) are recursively propagated over scans. For clarity, we give an outline of the step-by-step derivation of this process.

Assume the existence of prior detection PDF $p(H_{0,k}|\mathbf{Z}_{k-1}), p(H_{1,k}|\mathbf{Z}_{k-1})$ and prior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})$ at scan k. First, find prior target parameter state PDF $p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_{k-1})$ from (10). That and prior target absent probability $p(H_{0,k}|\mathbf{Z}_{k-1})$ constitute prior total target state PDF $p(\mathfrak{X}_k|\mathbf{Z}_{k-1})$ in (2) and (3).

Next, find posterior total target state PDF $p(\mathfrak{X}_k|\mathbf{Z}_k)$ in (2) and (3) via Bayes' formula but utilizing unnormalized PDFs and likelihood ratios (LRs) to simplify the development and implementation. Unnormalized posterior total target state PDF $p'(\mathfrak{X}_k|\mathbf{Z}_k)$ in (2) and (3) is

$$p'(H_{0,k}|\mathbf{Z}_k) = p_{LR}(\mathbf{z}_k|H_{0,k})p(H_{0,k}|\mathbf{Z}_{k-1}) \quad (11)$$

$$p'(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k) = p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_{k-1}) \quad (12)$$

The LRs in (11) and (12) are normalized likelihood functions (LFs) $p(\mathbf{z}_k|H_{0,k}), p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ from (4), (5).

$$p_{LR}(\mathbf{z}_k|H_{0,k}) = \frac{p(\mathbf{z}_k|H_{0,k})}{p(\mathbf{z}_k|H_{0,k})} = 1 \quad (13)$$

$$p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k) = \frac{p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)}{p(\mathbf{z}_k|H_{0,k})} \quad (14)$$

LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is the relative likelihood of target presence at target parameter state $\mathbf{x}_k$ with respect to target absence and contains all input data $\mathbf{z}_k$ detection and tracking information. Unnormalized posterior target absent probability $p'(H_{0,k}|\mathbf{Z}_k)$ and posterior target parameter state PDF $p'(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)$ in (11) and (12) are normalized, satisfying the identity in (7), to find posterior total target state PDF $p(\mathfrak{X}_k|\mathbf{Z}_k)$ in (2) and (3).

$$p(H_{0,k}|\mathbf{Z}_k) = \frac{p'(H_{0,k}|\mathbf{Z}_k)}{p'(H_{0,k}|\mathbf{Z}_k) + \int p'(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)\,d\mathbf{x}_k} \quad (15)$$

$$p(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k) = \frac{p'(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)}{p'(H_{0,k}|\mathbf{Z}_k) + \int p'(H_{1,k},\mathbf{x}_k|\mathbf{Z}_k)\,d\mathbf{x}_k} \quad (16)$$

Next, posterior detection PDF $p(H_{0,k}|\mathbf{Z}_k), p(H_{1,k}|\mathbf{Z}_k)$ and posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ are found using (6) and (9). To complete the cycle, prior detection PDF $p(H_{0,k+1}|\mathbf{Z}_k), p(H_{1,k+1}|\mathbf{Z}_k)$ and prior tracker PDF $p(\mathbf{x}_{k+1}|H_{1,k+1},\mathbf{Z}_k)$ are found for scan $k+1$ from these posterior PDFs based on their dynamic model.

This methodology depends on the equivalent reversible transformation from the detection and tracker PDFs to the total target state PDF in (2) and (3) with no loss of information, and vice versa.

2.3. Detection solution

Collapsing the steps in Section 2.2, the unnormalized posterior detection PDF Bayesian update is

$$p'(H_{0,k}|\mathbf{Z}_k) = p_{LR}(\mathbf{z}_k|H_{0,k})p(H_{0,k}|\mathbf{Z}_{k-1}) \quad (17)$$

$$p'(H_{1,k}|\mathbf{Z}_k) = p_{LR}(\mathbf{z}_k|H_{1,k})p(H_{1,k}|\mathbf{Z}_{k-1}) \quad (18)$$

Target absent LR $p_{LR}(\mathbf{z}_k|H_{0,k})$ in (17) is one from (13). Target present LR $p_{LR}(\mathbf{z}_k|H_{1,k})$ in (18) is

$$p_{LR}(\mathbf{z}_k|H_{1,k}) = \int p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})\,d\mathbf{x}_k$$
$$= \int p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)\,d\mathbf{x}_k \quad (19)$$

Target present LR $p_{LR}(\mathbf{z}_k|H_{1,k})$ in (19) is the average LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) weighted by prior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})$ in (26) and is also the total integral over unnormalized posterior tracker PDF $p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ in (24). For a detection only problem (known target parameter state $\mathbf{x}_k$), target present LR $p_{LR}(\mathbf{z}_k|H_{1,k})$ in (18) is LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14). Eq. (19) optimally reflects the uncertainty in target parameter state $\mathbf{x}_k$. Unnormalized posterior target detection PDF $p'(H_{0,k}|\mathbf{Z}_k), p'(H_{1,k}|\mathbf{Z}_k)$ in (17), (18) is normalized, satisfying the identity in (8).

$$p(H_{0,k}|\mathbf{Z}_k) = \frac{p'(H_{0,k}|\mathbf{Z}_k)}{p'(H_{0,k}|\mathbf{Z}_k) + p'(H_{1,k}|\mathbf{Z}_k)} \quad (20)$$

$$p(H_{1,k}|\mathbf{Z}_k) = \frac{p'(H_{1,k}|\mathbf{Z}_k)}{p'(H_{0,k}|\mathbf{Z}_k) + p'(H_{1,k}|\mathbf{Z}_k)} \quad (21)$$

Posterior target present probability $p(H_{1,k}|\mathbf{Z}_k)$ in (21) is the quantification of target presence confidence and is the optimal system level detection statistic. The tracking solution outputs in Section 2.4 are valid only for a sufficiently high target presence confidence.

Posterior target present probability $p(H_{1,k}|\mathbf{Z}_k)$ in (21) decreases, stays the same, or increases with respect to prior target present probability $p(H_{1,k}|\mathbf{Z}_{k-1})$ in (18) depending on whether target present LR $p_{LR}(\mathbf{z}_k|H_{1,k})$ in (18) is less than, equal to, or greater than one.

Prior target detection PDF $p(H_{0,k}|\mathbf{Z}_{k-1}), p(H_{1,k}|\mathbf{Z}_{k-1})$ in (17), (18) is known from the detection PDF Markov update in (22) or from initialization:

$$\begin{bmatrix} p(H_{0,k}|\mathbf{Z}_{k-1}) \\ p(H_{1,k}|\mathbf{Z}_{k-1}) \end{bmatrix} = \begin{bmatrix} p(H_{0,k}|H_{0,k-1}) & p(H_{0,k}|H_{1,k-1}) \\ p(H_{1,k}|H_{0,k-1}) & p(H_{1,k}|H_{1,k-1}) \end{bmatrix} \begin{bmatrix} p(H_{0,k-1}|\mathbf{Z}_{k-1}) \\ p(H_{1,k-1}|\mathbf{Z}_{k-1}) \end{bmatrix} \tag{22}$$

Posterior target detection PDF $p(H_{0,k-1}|\mathbf{Z}_{k-1})$, $p(H_{1,k-1}|\mathbf{Z}_{k-1})$ in (22) is known from (20), (21). Detection Markov transition probabilities $p(H_{i,k}|H_{j,k-1})$ (i, j=0,1) in (22) are the target detection state probabilities at scan k given a known target detection state at scan $k-1$. In Section 6, the detection Markov state transition probability matrix is binary symmetric:

$$\begin{bmatrix} p(H_{0,k}|H_{0,k-1}) & p(H_{0,k}|H_{1,k-1}) \\ p(H_{1,k}|H_{0,k-1}) & p(H_{1,k}|H_{1,k-1}) \end{bmatrix} = \begin{bmatrix} 1-q_{det} & q_{det} \\ q_{det} & 1-q_{det} \end{bmatrix} \tag{23}$$

The binary symmetric detection Markov state transition probability matrix in (23) drives prior target present probability $p(H_{1,k}|\mathbf{Z}_{k-1})$ in (22) toward the ambivalent target presence confidence of 0.5 with respect to posterior target present probability $p(H_{1,k-1}|\mathbf{Z}_{k-1})$ in (22).

The detection solution in this section requires input from the tracking solution in Section 2.4. In particular, prior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})$ in (26) is required to find target present LR $p_{LR}(\mathbf{z}_k|H_{1,k})$ in (19).

2.4. Tracking solution

Collapsing the steps in Section 2.2, the posterior tracker PDF Bayesian update is

$$p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k) = p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1}) \tag{24}$$

$$p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k) = \frac{p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)}{\int p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)\,d\mathbf{x}_k} \tag{25}$$

All tracking outputs are optimally computed from posterior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ in (25), which depends only on the shape of LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ due to normalization in (25). Unnormalized posterior tracker PDF $p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ in (24) also contains detection information (see (19)). Prior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})$ in (24) is known from the Markov update in (26) or from initialization.

$$p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1}) = \int p(\mathbf{x}_k|H_{1,k},\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|H_{1,k-1},\mathbf{Z}_{k-1})\,d\mathbf{x}_{k-1} \tag{26}$$

Posterior tracker PDF $p(\mathbf{x}_{k-1}|H_{1,k-1},\mathbf{Z}_{k-1})$ in (26) is known from (25). Tracker Markov transition PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{x}_{k-1})$ in (26) is the target parameter state $\mathbf{x}_k$ PDF given known target parameter state $\mathbf{x}_{k-1}$.

The tracking solution in this section assumes target presence over all scans and is "stand alone". In particular, this tracking only solution (as is also a Kalman filter) is independent of the detection solution in Section 2.3.

3. Measurement data single target detection and tracking solution

LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is required in (19) and (24). Tracker Markov transition PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{x}_{k-1})$ is required in (26). Both are found in Section 3.1 based on the standard measurement observation and target dynamic models. Then, the solution in Section 3.2 is a straightforward application of the general single target detection and tracking solutions in Sections 2.3 and 2.4.

3.1. Observation and dynamic models

LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is based on the standard uncertainty-of-origin measurement observation model [17–23]. We review this standard model briefly here. Input data $\mathbf{z}_k$ consists of M_k statistically independent measurement locations:

$$\mathbf{z}_k = [z_{k,1} \cdots z_{k,M_k}] \tag{27}$$

For target present, the target is either detected in the signal processing function with probability P_d or is not detected with probability $1-P_d$. For target present and detected, a single dimensionality J measurement $z_{k,m}$ in (27) is associated with the target. Its location is related to hypothesized target parameter state $\mathbf{x}_k$ through the standard linear observation model in (28) with random zero mean Gaussian observation error w_k having covariance matrix $\mathbf{C}_{\mathbf{z},k,m}$ resulting in the Gaussian PDF in (29) (see (A.1)):

$$z_{k,m} = \mathbf{H}_k\mathbf{x}_k + w_k \tag{28}$$

$$p(z_{k,m}|\mathbf{x}_k, \text{assoc}) = \mathcal{N}(z_{k,m}, \mathbf{H}_k\mathbf{x}_k, \mathbf{C}_{\mathbf{z},k,m}) \tag{29}$$

No association information is known. Therefore, all M_k measurements in (27) are equally likely to be the single associated measurement $z_{k,m}$ in (28) and (29). The other M_k-1 measurements in (27) are clutter. For target present and not detected or for target absent, all M_k measurements in (27) are clutter.

Further, the standard uncertainty-of-origin measurement observation model assumes that the number of clutter measurements is randomly Poisson distributed in (B.1) with parameter λ, the expected number, and that their locations are randomly uniformly i.i.d. over the entire observation space with constant value parameter ϖ (reciprocal of the entire volume). Then, each clutter measurement $z_{k,m}$ location PDF is $p_{unif}(z_{k,m};\varpi)$, and the joint clutter location PDF is their product.

The total joint clutter measurement PDF $p_{clut}(\mathbf{z}, M; \varpi, \lambda)$ (both locations $\mathbf{z}$ and number M) is

$$\begin{aligned} \underset{\mathbf{z}=[z_1\cdots z_M]}{p_{clut}(\mathbf{z}, M; \varpi, \lambda)} &= p_{unif}(\mathbf{z}; \varpi|M)p_{poi}(M;\lambda) \\ &= \left(\prod_{m=1}^{M} p_{unif}(z_m;\varpi)\right)p_{poi}(M;\lambda) \\ &= \varpi^M p_{poi}(M;\lambda) \end{aligned} \tag{30}$$

These assumptions imply a single sensor and a point target. Multiple sensor and distributed target solutions would be future work. The derivation for LR $p_{LR}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is taken from [27]. With this standard measurement observation model in hand, target present PDF $p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is

$$p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)=p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\text{target detected})\ P_d + p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\text{target not detected})\ (1-P_d) \tag{31}$$

Target present PDF $p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\text{target detected})$ in (31) is the total PDF over all M_k target association possibilities. The PDF of each possibility is the product of associated measurement location PDF $p(z_{k,m}|\mathbf{x}_k,\text{assoc})$ in (29) and the total joint clutter measurement PDF in (30) of the other M_k-1 measurements.

$$p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\ \text{target detected})=\sum_{m=1}^{M_k} p(z_{k,m}|\mathbf{x}_k,\text{assoc})$$

$$p_{\text{clut}}(\mathbf{z}_k^{(z_{k,m}\notin\mathbf{z}_k)},M_k-1;\varpi,\lambda)p(z_{k,m}=\text{assoc})$$

$$p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\ \text{target detected}) = \sum_{m=1}^{M_k}\mathcal{N}(z_{k,m},\mathbf{H}_k\mathbf{x}_k,\mathbf{C}_{\mathbf{z},k,m})\ (\varpi^{M_k-1}p_{\text{poi}}(M_k-1;\lambda))(M_k^{-1}) \tag{32}$$

Target present PDF $p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\text{target not detected})$ in (31) and target absent PDF $p(\mathbf{z}_k|H_{0,k})$ in (14) are all clutter in (30):

$$p(\mathbf{z}_k|H_{1,k},\mathbf{x}_k,\ \text{target not detected})=p(\mathbf{z}_k|H_{0,k}) = p_{\text{clut}}(\mathbf{z}_k,M_k;\varpi,\lambda)=\varpi^{M_k}p_{\text{poi}}(M_k;\lambda) \tag{33}$$

Consolidating, LR $p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (14) is

$$p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)=\sum_{m=1}^{M_k} a_{k,m}\mathcal{N}'(z_{k,m},\mathbf{H}_k\mathbf{x}_k,\mathbf{C}_{\mathbf{z},k,m})+b_k$$

$$a_{k,m}=P_d(\lambda\varpi)^{-1}((2\pi)^{J/2}|\mathbf{C}_{\mathbf{z},k,m}|^{1/2})^{-1} = P_d\beta^{-1}((2\pi)^{J/2}|\mathbf{C}_{\mathbf{z},k,m}|^{1/2})^{-1}$$

$$b_k=1-P_d \tag{34}$$

Function $\mathcal{N}'()$ in (34) is the unnormalized Gaussian PDF in (A.2). LR $p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (34) is multimodal (1 mode per measurement) and reflects the heuristic notion that the only measurement data information in a scan is that target presence is likely ($p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)>1$) at all measurement locations with equal likelihood but with a spread due to measurement location observation error and is unlikely but not impossible ($0<p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)<1$) far from any measurement location due to a possible target miss. The collective measurement data information over scans is distilled via the tracker PDF Bayesian and Markov updates in Section 3.2. LR $p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (34) can be expressed in terms of the more familiar clutter density β ($\beta=\lambda\varpi$) parameter. If any association or attribute information were known, that would be future work. The use of a LR is seen to be a simplification in (34).

Gaussian (see (A.1)) tracker Markov transition PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{x}_{k-1})$ in (36) is based on the standard target linear dynamic projection model in (35). Random target dynamic process noise $\mathbf{v}_{k-1}$ in (35) is zero mean Gaussian with covariance matrix $\mathbf{Q}_{k-1}$:

$$\mathbf{x}_k=\mathbf{F}_{k-1}\mathbf{x}_{k-1}+\mathbf{v}_{k-1} \tag{35}$$

$$p(\mathbf{x}_k|H_{1,k},\mathbf{x}_{k-1})=\mathcal{N}(\mathbf{x}_k,\mathbf{F}_{k-1}\mathbf{x}_{k-1},\mathbf{Q}_{k-1}) \tag{36}$$

$$\text{mean}(\mathbf{x}_k|\mathbf{Z}_{k-1})=\mathbf{F}_{k-1}\text{mean}(\mathbf{x}_{k-1}|\mathbf{Z}_{k-1})$$

$$\hat{\mathbf{x}}_{k|k-1}=\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} \tag{37}$$

$$\text{cov}(\mathbf{x}_k|\mathbf{Z}_{k-1})=\mathbf{F}_{k-1}\text{cov}(\mathbf{x}_{k-1}|\mathbf{Z}_{k-1})\mathbf{F}_{k-1}^{\mathrm{T}}+\mathbf{Q}_{k-1}$$

$$\mathbf{C}_{\mathbf{x},k|k-1}=\mathbf{F}_{k-1}\mathbf{C}_{\mathbf{x},k-1|k-1}\mathbf{F}_{k-1}^{\mathrm{T}}+\mathbf{Q}_{k-1} \tag{38}$$

A standard two state target position x_k and rate $\dot{x}_k$ dynamic model with scan length T [22] in (39) was selected for Section 6. Rate of change per scan Δx_k ($\Delta x_k=T\dot{x}_k$) was actually used.

$$\mathbf{x}_k=\begin{bmatrix}x_k\\ \dot{x}_k\end{bmatrix},\quad \mathbf{F}_k=\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}$$

$$\mathbf{Q}_k=(q_{trk}T)\begin{bmatrix}T^2/3 & T/2\\ T/2 & 1\end{bmatrix} \tag{39}$$

3.2. Solution

With the definition of the standard statistical models on measurement input data and target kinematics given above, the repeating tracker PDF canonical form for both the Bayesian update in (44) and the Markov update in (47) is a Gaussian mixture (see Appendix C) with each PDF $\mathcal{N}()$ from (A.1) a Gaussian "mode".

Unnormalized posterior tracker PDF $p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)$ in (40) is found from (24). LR $p_{\text{LR}}(\mathbf{z}_k|H_{1,k},\mathbf{x}_k)$ in (24) is known from (34). Prior tracker PDF $p(\mathbf{x}_k|H_{1,k},\mathbf{Z}_{k-1})$ in (24) is known from the tracker PDF Markov update in (47) or from initialization:

$$p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)=\left(\sum_{m=1}^{M_k}a_{k,m}\mathcal{N}'(z_{k,m},\mathbf{H}_k\mathbf{x}_k,\mathbf{C}_{\mathbf{z},k,m})+b_k\right) \times\left(\sum_{n=1}^{N_{k|k-1}}p_{k|k-1,n}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n})\right) \tag{40}$$

Expand (40):

$$p'(\mathbf{x}_k|H_{1,k},\mathbf{Z}_k)=\sum_{n=1}^{N_{k|k-1}}b_kp_{k|k-1,n}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n}) +\sum_{m=1}^{M_k}\sum_{n=1}^{N_{k|k-1}}a_{k,m}p_{k|k-1,n}\mathcal{N}'(z_{k,m},\mathbf{H}_k\mathbf{x}_k,\mathbf{C}_{\mathbf{z},k,m}) \times\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n}) \tag{41}$$

From (A.3), each measurement m and prior Gaussian mode n product in (41) can be expressed as a posterior Gaussian mode with Gaussian product factor $\rho_{k,m,n}$, which plays a major role:

$$\mathcal{N}'(z_{k,m},\mathbf{H}_k\mathbf{x}_k,\mathbf{C}_{\mathbf{z},k,m})\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n}) =\rho_{k,m,n}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k,m,n},\mathbf{C}_{\mathbf{x},k|k,m,n}) \tag{42}$$

Substitute Gaussian product $\mathcal{N}'()\mathcal{N}()$ from (42) into (41) and collect all terms in (41) into a single summation over all posterior Gaussian modes n', each with a single weight $p'_{k|k,n'}$:

$$\mathrm{p}'(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)=\sum_{n'=1}^{N_{k|k}} p'_{k|k,n'}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k,n'},\mathbf{C}_{\mathbf{x},k|k,n'})$$

$$N_{k|k}=N_{k|k-1}+M_kN_{k|k-1}$$

$$p'_{k|k,n'}=(a_{k,m}\rho_{k,m,n})(p_{k|k-1,n}) \quad \text{or} \quad (b_k)(p_{k|k-1,n}) \tag{43}$$

Unnormalized posterior tracker PDF $\mathrm{p}'(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)$ in (43) is normalized following (25):

$$\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)=\sum_{n=1}^{N_{k|k}} p_{k|k,n}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k,n},\mathbf{C}_{\mathbf{x},k|k,n})$$

$$p_{k|k,n}=p'_{k|k,n}/\sum_{n'=1}^{N_{k|k}} p'_{k|k,n'}. \tag{44}$$

Posterior tracker PDF $\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)$ in (44) reduces to a normalization of unnormalized Gaussian mode weights $p'_{k|k,n'}$ in (43). Prior tracker PDF $\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_{k-1})$ in (45) is found from (26). Posterior tracker PDF $\mathrm{p}(\mathbf{x}_{k-1}|\mathrm{H}_{1,k-1},\mathbf{Z}_{k-1})$ in (26) is known from the tracker PDF Bayesian update in (44). Tracker Markov transition PDF $\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{x}_{k-1})$ in (26) is known from (36):

$$\begin{aligned}\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_{k-1})&=\sum_{n=1}^{N_{k-1|k-1}} p_{k-1|k-1,n}\\&\times\left\{\int\mathcal{N}(\mathbf{x}_k,\mathbf{F}_{k-1}\mathbf{x}_{k-1},\mathbf{Q}_{k-1})\mathcal{N}(\mathbf{x}_{k-1},\hat{\mathbf{x}}_{k-1|k-1,n},\mathbf{C}_{\mathbf{x},k-1|k-1,n})\,\mathrm{d}\mathbf{x}_{k-1}\right\}\end{aligned} \tag{45}$$

The tracker PDF Markov update in (45) reduces to a Markov update for each Gaussian mode n (in braces), which is a Gaussian PDF (see (A.1)) with mean $\hat{\mathbf{x}}_{k|k-1,n}$ and covariance matrix $\mathbf{C}_{\mathbf{x},k|k-1,n}$ from (37), (38) [22]:

$$\begin{aligned}&\int\mathcal{N}(\mathbf{x}_k,\mathbf{F}_{k-1}\mathbf{x}_{k-1},\mathbf{Q}_{k-1})\mathcal{N}(\mathbf{x}_{k-1},\hat{\mathbf{x}}_{k-1|k-1,n},\mathbf{C}_{\mathbf{x},k-1|k-1,n})\,\mathrm{d}\mathbf{x}_{k-1}\\&=\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n})\end{aligned} \tag{46}$$

Substitute (46) into (45). Gaussian mode weights $p_{k-1|k-1,n}$ are unchanged. Then, prior tracker PDF $\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_{k-1})$ in (45) is

$$\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_{k-1})=\sum_{n=1}^{N_{k|k-1}} p_{k|k-1,n}\mathcal{N}(\mathbf{x}_k,\hat{\mathbf{x}}_{k|k-1,n},\mathbf{C}_{\mathbf{x},k|k-1,n})$$

$$N_{k|k-1}=N_{k-1|k-1},\quad p_{k|k-1,n}=p_{k-1|k-1,n} \tag{47}$$

Target present LR $\mathrm{p}_{\mathrm{LR}}(\mathbf{z}_k|\mathrm{H}_{1,k})$ in (19) is required in the detection solution in Section 2.3. Substitute unnormalized posterior tracker PDF $\mathrm{p}'(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)$ from (43) into (19).

$$\mathrm{p}_{\mathrm{LR}}(\mathbf{z}_k|\mathrm{H}_{1,k})=\sum_{n=1}^{N_{k|k}} p'_{k|k,n} \tag{48}$$

With target present LR $\mathrm{p}_{\mathrm{LR}}(\mathbf{z}_k|\mathrm{H}_{1,k})$ known from (48), the detection PDF Bayesian update in (17), (18), (20), and (21) and, then, the detection PDF Markov update in (22) are performed. The use of unnormalized PDFs to obtain (48) is seen to be a simplification.

This solution is correct based on the required compliance with the general single target detection and tracking solution in Section 2 with LR $\mathrm{p}_{\mathrm{LR}}(\mathbf{z}_k|\mathrm{H}_{1,k},\mathbf{x}_k)$ from (34) and tracker Markov transition PDF $\mathrm{p}(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{x}_{k-1})$ from (36), which themselves are the correct solutions based on their standard models.

The Kalman filter solution (single Gaussian mode, see Appendix A) and Bayesian conditional PDF (unimodal Gaussian shape) are, respectively, special cases of this solution and LR $\mathrm{p}_{\mathrm{LR}}(\mathbf{z}_k|\mathrm{H}_{1,k},\mathbf{x}_k)$ in (34). The latter are extensions for the additional uncertainty-of-origin conditions in Section 1. Only linear, Gaussian observation and dynamic models are considered. Nonlinear, non-Gaussian models would be future work.

The Gaussian modes are the fundamental objects. They are computed independently of any single target solution, or detector-tracker; are the only input required for any detector-tracker's recursive computation at a scan; and only require a recursive computation at a scan with measurements as the input.

A Gaussian mode is the Kalman filter solution for a single target measurement path over scans (1 per scan, see Appendix A). A detector-tracker has an irresistible proclivity to seek the strongest target path in its available measurement input data by virtue of integration in the tracker PDF Bayesian update. I.e., the Gaussian mode weights unimodally peak at the correct Gaussian mode with the others negligible. Then, the solution depends only on a subset of the measurements: those contributing to the dominant, or peak, Gaussian modes, which may be viewed as the correctly associated measurements. This property is demonstrated in Section 6.

4. Measurement data multiple target detection and tracking solution

For L potential targets with joint total multiple target state $\mathfrak{x}_k^{\mathfrak{L}}$, joint posterior total multiple target state PDF $\mathrm{p}(\mathfrak{x}_k^{\mathfrak{L}}|\mathbf{Z}_k)$ is

$$\mathrm{p}(\mathfrak{x}_k^{\mathfrak{L}}|\mathbf{Z}_k)=\mathrm{p}(\mathfrak{x}_k^{(1)},\mathfrak{x}_k^{(2)},\ldots,\mathfrak{x}_k^{(L)}|\mathbf{Z}_k) \tag{49}$$

Each total single target state $\mathfrak{x}_k^{(l)}$ in (49) is a total single target state $\mathfrak{x}_k$ in Section 2.1. Each posterior total single target state PDF $\mathrm{p}(\mathfrak{x}_k^{(l)}|\mathbf{Z}_k)$ in (2) and (3) can be found as a marginal of (49). The straightforward extension of the posterior total single target state PDF $\mathrm{p}(\mathfrak{x}_k|\mathbf{Z}_k)$ solution in Sections 2 and 3 to find joint posterior total multiple target state PDF $\mathrm{p}(\mathfrak{x}_k^{\mathfrak{L}}|\mathbf{Z}_k)$ in (49) is, in general, computationally infeasible.

Therefore, the multiple target approach exploits the detector-tracker properties in Section 3.2. If the dominant Gaussian modes of each detector-tracker can be excluded from all other detector-trackers, their solutions are mutually orthogonal and thus independent. The detector-trackers do not "share" Gaussian modes. Only one Gaussian mode weight can be "high" over all detector-trackers. Then, joint posterior total multiple target state PDF $\mathrm{p}(\mathfrak{x}_k^{\mathfrak{L}}|\mathbf{Z}_k)$ in (49) can be factored as the product of all these resulting posterior total single target state PDFs $\mathrm{p}(\mathfrak{x}_k^{(l)}|\mathbf{Z}_k)$:

$$\mathrm{p}(\mathfrak{x}_k^{\mathfrak{L}}|\mathbf{Z}_k)=\prod_{l=1}^{L}\mathrm{p}(\mathfrak{x}_k^{(l)}|\mathbf{Z}_k) \tag{50}$$

The multiple target approach is the realization of (50). Consider the single target detector-tracker solution structure in Section 3.2. Unnormalized posterior tracker PDF $\mathrm{p}'(\mathbf{x}_k|\mathrm{H}_{1,k},\mathbf{Z}_k)$ in (43) is the only input required for the remaining detection and tracking Bayesian and Markov PDF updates. Then, unnormalized posterior tracker PDFs $\mathrm{p}''(\mathbf{x}_k^{(l)}|\mathrm{H}_{1,k}^{(l)},\mathbf{Z}_k)$ are found as the marginals of the joint multiple target unnormalized posterior tracker PDF for use in those updates.

For the two target case, the marginal solution for unnormalized posterior tracker 1 PDF $\mathrm{p}''(\mathbf{x}_k^{(1)}|\mathrm{H}_{1,k}^{(1)},\mathbf{Z}_k)$ is

$$\mathrm{p}''(\mathbf{x}_k^{(1)}|\mathrm{H}_{1,k}^{(1)},\mathbf{Z}_k) = \alpha\int\left(\sum_{n_1=1}^{N_{k|k}} p'^{(1)}_{k|k,n_1}\mathcal{N}(\mathbf{x}_k^{(1)},\hat{\mathbf{x}}_{k|k,n_1},\mathbf{C}_{\mathbf{x},k|k,n_1})\right)\times\left(\sum_{n_2=1}^{N_{k|k}} p'^{(2)}_{k|k,n_2}\mathcal{N}(\mathbf{x}_k^{(2)},\hat{\mathbf{x}}_{k|k,n_2},\mathbf{C}_{\mathbf{x},k|k,n_2})\right)\mathrm{f}(n_1,n_2)\,\mathrm{d}\mathbf{x}_k^{(2)}$$

$$\mathrm{f}(n_1,n_2)=1-\delta(n_1-n_2)$$

$$\alpha=\left(\int \mathrm{p}'(\mathbf{x}_k^{(2)}|\mathrm{H}_{1,k}^{(2)},\mathbf{Z}_k)\,\mathrm{d}\mathbf{x}_k^{(2)}\right)^{-1}=\left(\sum_{n_2=1}^{N_{k|k}} p'^{(2)}_{k|k,n_2}\right)^{-1} \tag{51}$$

Each factor in parentheses in (51) is the single target unnormalized posterior tracker PDF in (43). Their product is the joint 2 target unnormalized posterior tracker PDF based on an independence assumption (the desired case) and since no joint information is available. The "non-sharing" of Gaussian modes is not an inherent property as is the strongest target path seeking behavior. For (51) expanded, it is equivalent to neglecting terms with the same Gaussian modes (n_1=n_2). I.e., both detector-trackers cannot exist at the same Gaussian mode. This is enforced by boundary condition f(n_1,n_2) (f(n_1,n_2)=0 for n_1=n_2 and 1 otherwise) in (51), which is additional prior information.

The unnormalized posterior tracker PDFs both individually and collectively in their joint PDF are improper PDFs (do not integrate to one). Therefore, a scale factor α is required in (51). The rationale for scale factor α is that unnormalized posterior tracker 1 PDF $\mathrm{p}''(\mathbf{x}_k^{(1)}|\mathrm{H}_{1,k}^{(1)},\mathbf{Z}_k)$ in (51) must reduce to the single target case in (43) ($\mathrm{p}'(\mathbf{x}_k^{(1)}|\mathrm{H}_{1,k}^{(1)},\mathbf{Z}_k)$) for independent operation (f(n_1,n_2)=1), preserving the correct scale for detection.

Find unnormalized posterior tracker 1 PDF $\mathrm{p}''(\mathbf{x}_k^{(1)}|\mathrm{H}_{1,k}^{(1)},\mathbf{Z}_k)$ in (51) and generalize to unnormalized posterior tracker l PDF $\mathrm{p}''(\mathbf{x}_k^{(l)}|\mathrm{H}_{1,k}^{(l)},\mathbf{Z}_k)$ $(l=1,\ldots,L)$ for the L target case.

$$\mathrm{p}''(\mathbf{x}_k^{(l)}|\mathrm{H}_{1,k}^{(l)},\mathbf{Z}_k)=\sum_{n=1}^{N_{k|k}} p''^{(l)}_{k|k,n}\mathcal{N}(\mathbf{x}_k^{(l)},\hat{\mathbf{x}}_{k|k,n},\mathbf{C}_{\mathbf{x},k|k,n})$$

$$p''^{(l)}_{k|k,n}=p'^{(l)}_{k|k,n}\prod_{\substack{l'=1\\ l'\neq l}}^{L} q^{(l')}_{k|k,n}$$

$$q^{(l')}_{k|k,n}=1-p'^{(l')}_{k|k,n}\Big/\sum_{n'=1}^{N_{k|k}} p'^{(l')}_{k|k,n'}. \tag{52}$$

Gaussian modes $\mathcal{N}(\mathbf{x}_k^{(l)},\hat{\mathbf{x}}_{k|k,n},\mathbf{C}_{\mathbf{x},k|k,n})$ and their weights $p'^{(l)}_{k|k,n}$ in (52) are known from (43). Using unnormalized posterior tracker l PDF $\mathrm{p}''(\mathbf{x}_k^{(l)}|\mathrm{H}_{1,k}^{(l)},\mathbf{Z}_k)$ in (52), posterior detection PDF $\mathrm{p}(\mathrm{H}_{0,k}^{(l)}|\mathbf{Z}_k),\mathrm{p}(\mathrm{H}_{1,k}^{(l)}|\mathbf{Z}_k)$ and tracker PDF $\mathrm{p}(\mathbf{x}_k^{(l)}|\mathrm{H}_{1,k}^{(l)},\mathbf{Z}_k)$ for detector-tracker l are computed from the remaining detection and tracker PDF Bayesian and Markov updates in Section 3.2.

Unnormalized posterior Gaussian mode weight $p''^{(l)}_{k|k,n}$ in (52) is decreased with respect to unnormalized posterior Gaussian mode weight $p'^{(l)}_{k|k,n}$ in (43) for any other weight $p'^{(l')}_{k|k,n}$ "high" through multiplication by its "inverse" weight $q^{(l')}_{k|k,n}$ $(0\le q^{(l')}_{k|k,n}\le 1)$, which is "low". Otherwise, there is little effect ($q^{(l')}_{k|k,n}\approx 1$). This operation directly results from boundary condition f() in (51) and "partitions" Gaussian modes and, thus, measurement data over detector-trackers. It may be viewed as an indirect association of measurement data over detector-trackers and is the counterpart to the explicit measurement association hypotheses in JPDA and MHT.

For steady-state, the detector-trackers closely achieve the mutual orthogonality in (50) and are essentially optimal. The marginalization suppresses any tendency for the Gaussian modes to commingle over the detector-trackers. For initialization or acquisition, the multiple target solution is driven to this ideal state. Only one detector-tracker will "win" at each target path. This property is demonstrated in Section 6.

The Gaussian modes are a common resource over all detector-trackers and are a self-propagating basis set for all the posterior tracker PDFs yielding simplicity and efficiency in implementation. They play an analogous role to a fixed grid [1,7,28] and a particle filter (adaptive grid) [29–31] point basis set representation of the posterior tracker PDF but yield a richer, less sparse representation of the target parameter state space.

From this viewpoint, the tracking solutions are the computation of the Gaussian mode weights, which may be regarded as the probabilities of each Gaussian mode being the correct solution and, thus, are a " soft" selection over the Gaussian modes and thus also over all single target measurement path Kalman filter solutions. The Gaussian mode weights are the only differentiation over detector-trackers. This viewpoint is demonstrated in Section 6.

The detector-tracker global convergence to the strongest target path enables the desirable detector-trackers' global target search and acquisition but necessitates selective data restriction through the inverse weights to prevent the undesirable detector-trackers' collocation for a multiple target problem. This property is critical to the operation in Section 6 and is demonstrated. Problems where targets can be collocated and thus can "share" Gaussian modes (e.g., bearing observation space with finite resolvability) would be future work.

5. Gaussian modes

Gaussian mode n discrete target absent, present states at scan k are hypotheses $\mathrm{H}_{0,k}^{(n)}$, $\mathrm{H}_{1,k}^{(n)}$. Their solution is the same as the single target detection solution in Section 3.2 but with the computation restricted to a single Gaussian mode with unity weight. Then, their posterior probabilities are recursively propagated in the detection PDF Bayesian update in (17), (18), (20), and (21) and the detection PDF Markov update in (22) with target absent, present LRs $p_{LR}(\mathbf{z}_k|H_{0,k}^{(n)}), p_{LR}(\mathbf{z}_k|H_{1,k}^{(n)})$ from (53):

$$p_{LR}(\mathbf{z}_k|H_{0,k}^{(n)}) = 1$$

$$p_{LR}(\mathbf{z}_k|H_{1,k}^{(n)}) = a_{k,m}\rho_{k,m,n} \text{ or } b_k \tag{53}$$

An issue is the growth in Gaussian modes from the tracker PDF Bayesian update in (43), similar to the growth in association hypotheses for MHT. Gaussian mode reduction is required. In Section 6, two criteria were applied. The first is weak Gaussian modes based on small posterior Gaussian mode target present probability. This causes the termination of weak Gaussian mode strings and acts as a gate for measurements with respect to Gaussian modes. The second is redundant Gaussian modes based on " closeness" using the Mahalanobis distance. This eliminates the weaker of two Gaussian mode strings converging to the same solution due to common input measurements and prevents initialization of a measurement near a Gaussian mode.

A distinct Gaussian mode is the "null", or 1st, Gaussian mode and is the representation of minimal target parameter state $\mathbf{x}_k$ knowledge. It has a large covariance matrix $\mathbf{C}_{\mathbf{x},k|k,1}$ and a nominal mean $\hat{\mathbf{x}}_{k|k,1}$ spanning the entire target parameter state $\mathbf{x}_k$ space and has significant posterior Gaussian mode weight $p_{k|k,1}^{(l)}$ in (44) for a detector-tracker l only for no good match to any other Gaussian mode. The null Gaussian mode is not included in the inverse weighting operation in (52). I.e., it can be shared by detector-trackers. It is included in the unnormalized tracker PDF Bayesian update in (40), which may be viewed as an initialization for a measurement.

6. Simulation results

The purpose of the simulation is to demonstrate performance where clutter is an issue and operation. Scalar measurements were directly simulated in a generic scalar x position observation space following the standard models in Section 3.1. An example detector-tracker management procedure is described and is utilized.

Truth for four target paths is in Fig. 1. The position observation range is {0.0,100.0} with 200 total scans. The targets have staggered start scans, and all end at scan 170. Each target path is piecewise constant rate and has a 0.1 scan rate step change every 20 scans with maximum 0.5. All target measurements have 0.9 probability of detection and zero mean Gaussian position observation error with standard deviation 0.05. Prior maximum rate information is available through a high variance 0.0 pseudo-rate observation in each measurement. Clutter measurement positions are uniformly i.i.d. {0.0,100.0} with the number Poisson distributed ($\lambda = 85.0$) in (B.1). A resolution (minimum measurement spacing) of 0.3 is enforced causing a reduction to an average of 67.3. The measurement input data is in Fig. 2. The true target tracks (with misses and errors) can be identified.

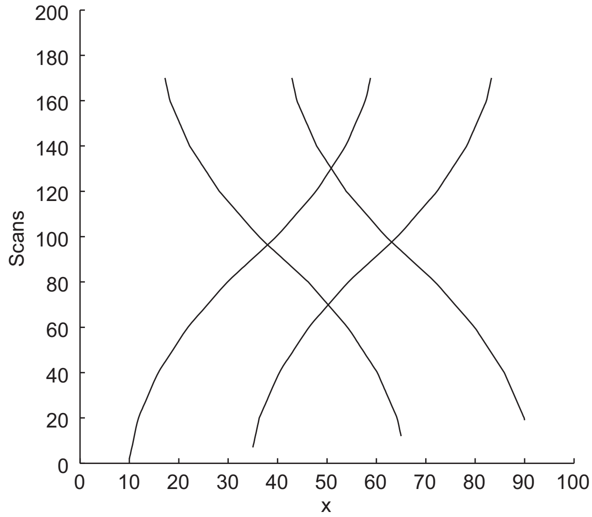

Fig. 1. Target truth.

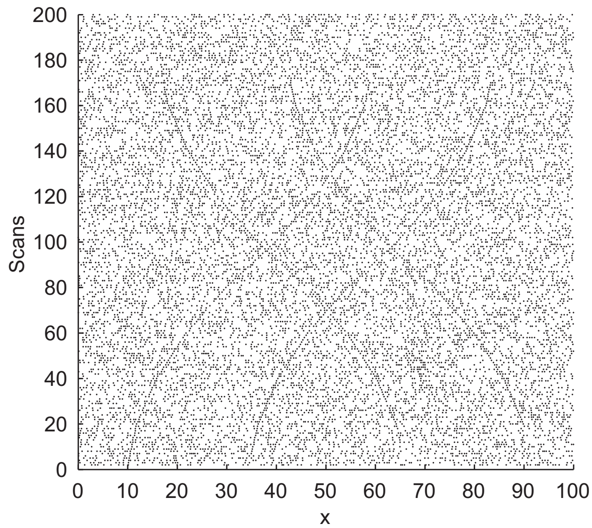

Fig. 2. Simulated measurement data.

The Gaussian mode solution is demonstrated in Fig. 3. The size of each point is modulated by the posterior Gaussian mode target present probability in Section 5. True target tracks and spurious tracks can be identified. The number of Gaussian modes is on the order of the number of measurements demonstrating the Gaussian mode reduction in Section 5, which does not adversely affect performance.

The detection statistic outputs in Fig. 4 are posterior target present probability $p(H_{1,k}^{(l)}|\mathbf{Z}_k)$ in (21). Three tracking outputs are shown in Figs. 5–14. The first demonstrates the detector-tracker "selection" process over posterior Gaussian modes by the size of each modulated by its weight $p_{k|k,n}^{(l)}$ in (44) similar to Fig. 3. The second is the posterior tracker PDF means (MMSE estimates) from (C.1). The third is the target truth.

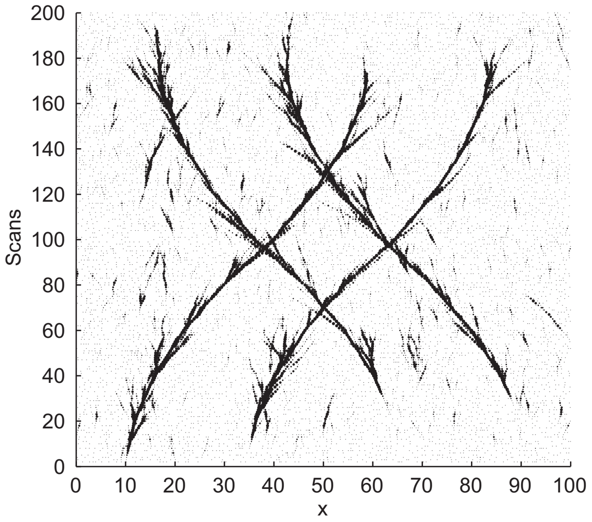

Fig. 3. Gaussian modes.

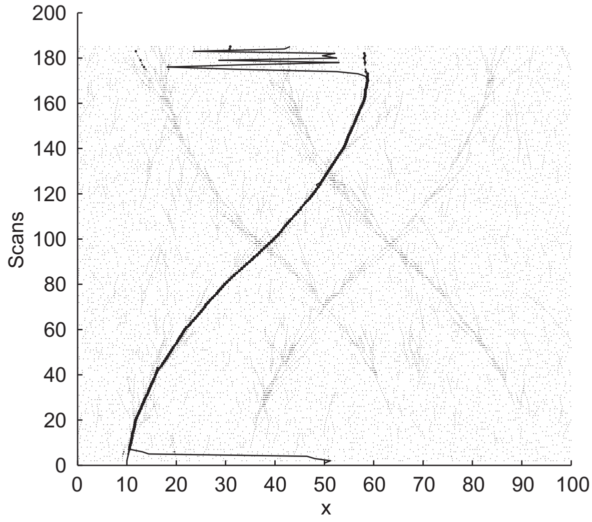

Fig. 5. Detector-tracker 1 position (x) output.

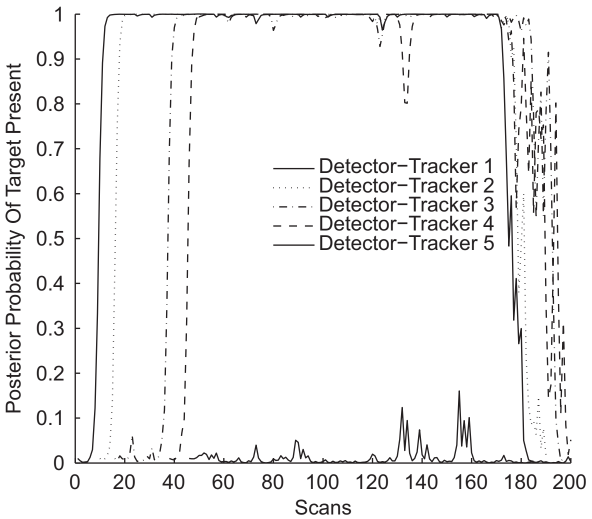

Fig. 4. Detection statistics.

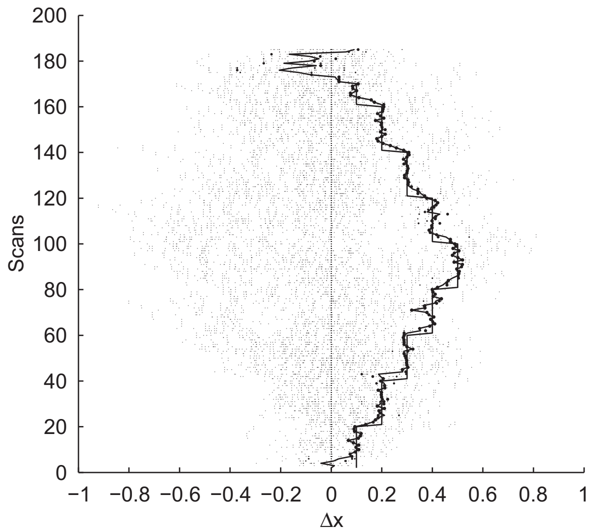

Fig. 6. Detector-tracker 1 rate (Δx) output.

No target kinematic knowledge is required for initialization of a detector-tracker (uniform Gaussian mode weights), and it is unconfirmed. Its target presence confidence is low, and its tracking outputs are not valid. An unconfirmed detector-tracker is raised to tentative if its detection statistics rise above a threshold. A target detection is not made, but an unconfirmed detector-tracker is initiated to search for a "new" target. A tentative detector-tracker is raised to confirmed if its detection statistics rise above a higher threshold. A target detection is declared, and its tracking outputs are now valid. A confirmed detector-tracker is lowered to tentative if its detection statistics fall below a threshold and is dropped for a lower threshold. At any scan, there is only one unconfirmed detector-tracker, globally searching for a new target over the entire target parameter state space. A multitude of unconfirmed Kalman filter based trackers is not required to perform this function.

At start-up, the system is initialized to unconfirmed detector-tracker 1 and one null Gaussian mode (see Section 5). Detector-tracker 1 posterior Gaussian mode weights $p^{(1)}_{k|k,n}$ in (44) and, thus, posterior tracker 1 PDF mean converge to target 1, the strongest initial target path. The "background" Gaussian modes in Figs. 5–14 have low weight. The background persistent target rate of zero is due to the measurement pseudo-rate observation. Detector-tracker 1 detection statistics converge to nearly one indicating high target presence confidence.

Detector-tracker 1 initiates detector-tracker 2, which similarly detects, acquires, and follows target 2, the strongest available target path at its inception. Detector-tracker 1 "denies" target 1 to detector-tracker 2 and all

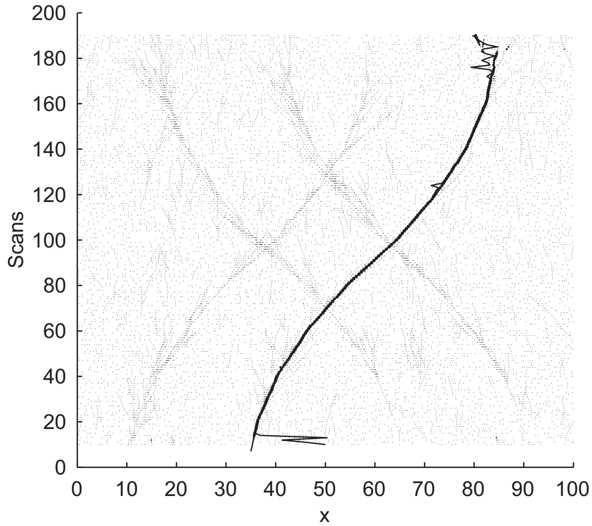

Fig. 7. Detector-tracker 2 position (x) output.

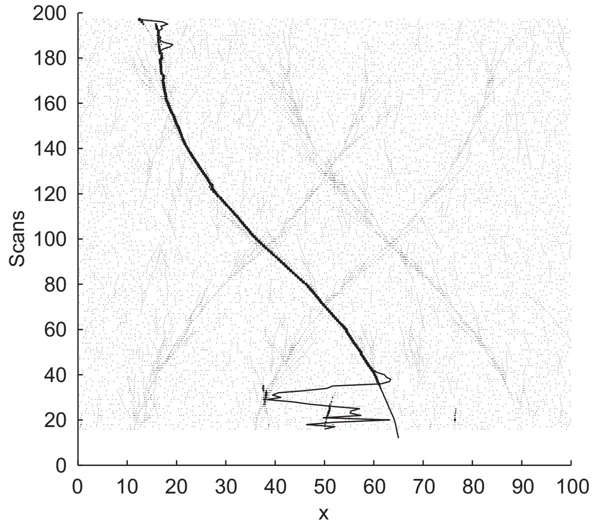

Fig. 9. Detector-tracker 3 position (x) output.

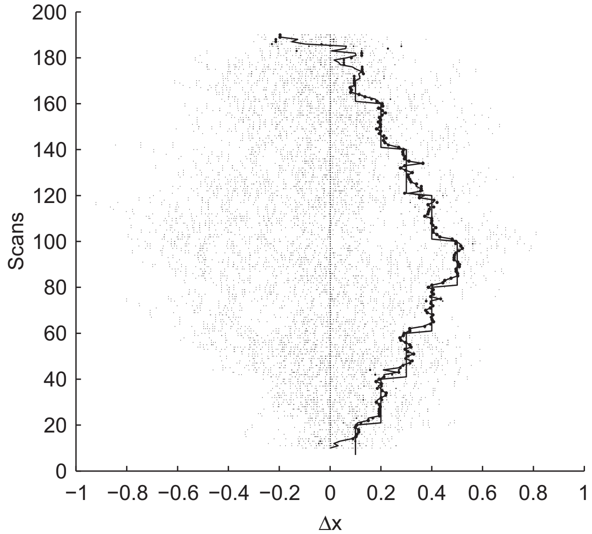

Fig. 8. Detector-tracker 2 rate (Δx) output.

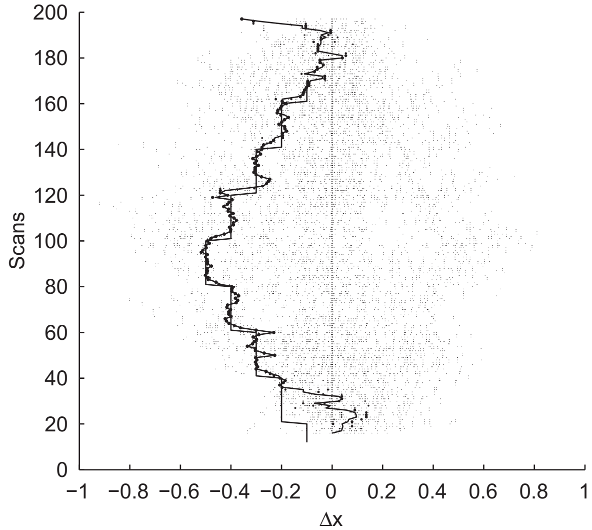

Fig. 10. Detector-tracker 3 rate (Δx) output.

other subsequently initiated detector-trackers through its inverse weights $q_{k|k,n}^{(1)}$ in (52). The same events transpire, and this cycle is repeated. Detector-trackers 3 and 4 detect, acquire, and follow targets 3 and 4. After the targets disappear, detector-trackers 1, 2, 3, 4 tracking outputs diverge, and their detection statistics fall. Detector-trackers 1, 2, 3 are dropped by scenario end with detector-tracker 4 on the verge.

Detector-tracker 5 is the exception. Its tracking outputs in Figs. 13 and 14 are a random walk over the entire target parameter state space. This is the correct behavior since targets 1, 2, 3, 4 are "taken" and none are left. This behavior further demonstrates the detector-tracker global target search and acquisition and the inverse weighting operation, which enforces non-sharing of Gaussian modes and thus non-collocation over detector-trackers. Occasionally, detector-tracker 5 temporarily acquires a "clutter track" but its detection statistics in Fig. 4 correctly remain low, and it remains unconfirmed. Any detections would have been false alarms.

7. Summary and conclusions

The real time multitarget detection and tracking problem (including uncertainty-of-origin issues) is sufficiently developed so that there exists standard accepted models and solution methodology. This paper is grounded in these models and methodology even though the solution in totality is novel. The standard models and methodology are locally well defined. However, the weaving into an optimal system level solution has not been achieved until now.

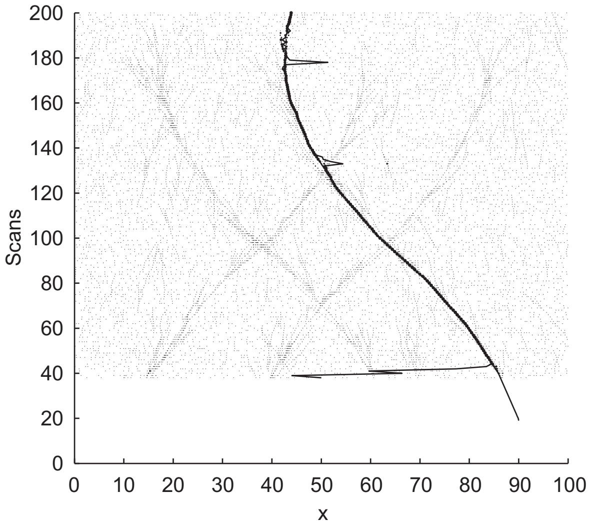

Fig. 11. Detector-tracker 4 position (x) output.

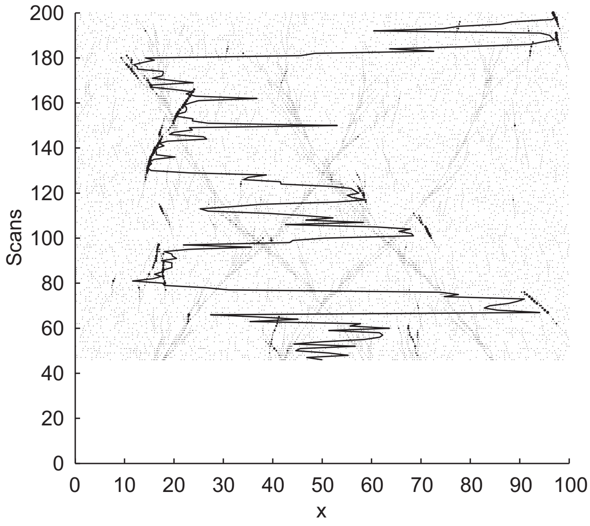

Fig. 13. Detector-tracker 5 position (x) output.

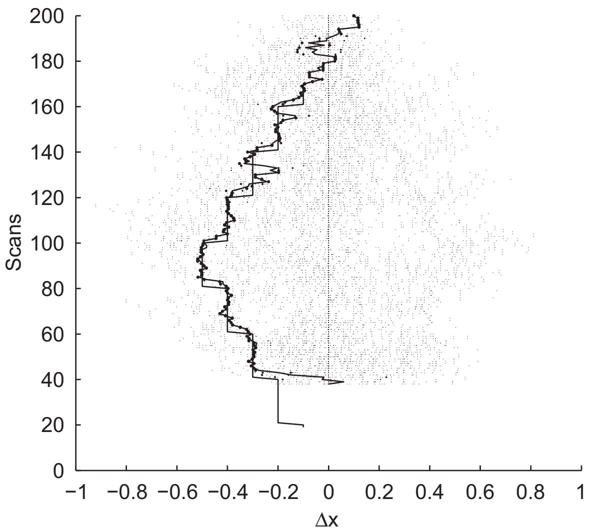

Fig. 12. Detector-tracker 4 rate (Δx) output.

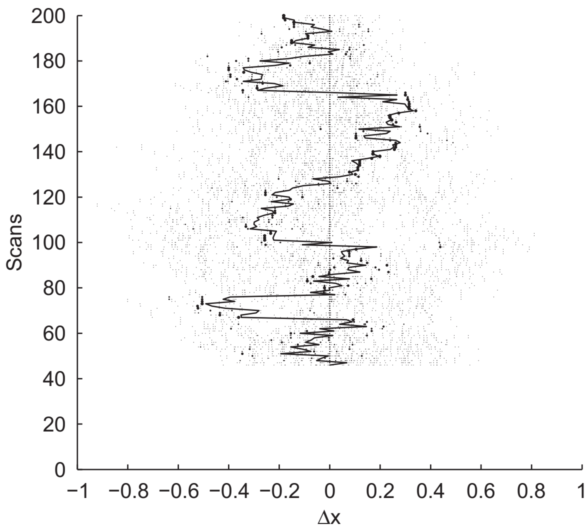

Fig. 14. Detector-tracker 5 rate (Δx) output.

This paper has derived the optimal Bayes–Markov filter solution for the multiple target uncertainty-of-origin problem. We have shown that under standard linear, Gaussian assumptions the posterior PDF is a repeating Gaussian mixture with a common basis set over targets and have provided the analytic formulation for this Gaussian mixture. In addition, we have presented an efficient novel computational approach where we capture the relevant Gaussian modes by removing weak or redundant Gaussian modes. This process is analogous to pruning hypotheses in MHT.

A full comparison to MHT has not been performed in this paper. However, the proposed technique has advantages over MHT for the following reasons. First, our method is based on a rigorous analytic formulation of the recursive posterior target PDF, which shows that the correct posterior PDF is a Gaussian mixture. In contrast, standard techniques assume the posterior PDF is Gaussian and then work to decide which measurement to give to each standard Kalman filter. Second, we are able to select which modes (hypotheses) to remove from our catalog in an optimal manner—e.g., by removing hypothesis that are least likely to be correct. Standard techniques employ various heuristics for this requirement rather than precisely computing the hypothesis probability. Finally, the solution optimally jointly estimates the target existence probability and the target parameter state, fully capturing the coupling of uncertainty in these estimates. In contrast, other approaches treat these problems separately.

There are several avenues for future work. One avenue is to relax the assumption of (50). I.e., rather than treating the multiple target problem as multiple coupled single target problems, we capture the coupling between targets in a joint manner. A potential approach is to factor (49) into clusters of single targets, target pairs, target triplets, and so on, adaptively selected based on target proximity and then treat the clusters jointly along the lines of [30]. A second avenue is to investigate the effect of non-thresholded measurements on the form of the recursive PDF. Finally, we will investigate extending this solution to (mildly) nonlinear systems where the extended [22] or unscented [32] Kalman filter is used as the base PDF rather than the standard Kalman filter.

Acknowledgments

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Appendix A. Gaussian PDFs and product

Gaussian PDF $\mathcal{N}()$ is (A.1) with dimensionality J random variable $\mathbf{x}$, mean $\overline{\mathbf{x}}$, and covariance matrix $\mathbf{C}$.

$$\mathcal{N}(\mathbf{x},\overline{\mathbf{x}},\mathbf{C}) = ((2\pi)^{J/2}|\mathbf{C}|^{1/2})^{-1}\exp(-0.5(\mathbf{x}-\overline{\mathbf{x}})^{\mathsf{T}}\mathbf{C}^{-1}(\mathbf{x}-\overline{\mathbf{x}})) \tag{A.1}$$

Unnormalized Gaussian PDF $\mathcal{N}'()$ (max=1) in (A.2) is (A.1) with only the exponent factor:

$$\mathcal{N}'(\mathbf{x},\overline{\mathbf{x}},\mathbf{C}) = \exp(-0.5(\mathbf{x}-\overline{\mathbf{x}})^{\mathsf{T}}\mathbf{C}^{-1}(\mathbf{x}-\overline{\mathbf{x}})) \tag{A.2}$$

The Gaussian product $\mathcal{N}'(\mathbf{z},\mathbf{Hx},\mathbf{C_z})\mathcal{N}(\mathbf{x},\overline{\mathbf{x}},\mathbf{C}_{\overline{\mathbf{x}}})$ can be identically expressed as a Gaussian PDF with a factor:

$$\mathcal{N}'(\mathbf{z},\mathbf{Hx},\mathbf{C_z})\mathcal{N}(\mathbf{x},\overline{\mathbf{x}},\mathbf{C}_{\overline{\mathbf{x}}}) = \rho\mathcal{N}(\mathbf{x},\hat{\mathbf{x}},\mathbf{C}_{\hat{\mathbf{x}}})$$

$$\mathbf{C}_{\hat{\mathbf{x}}}^{-1} = \mathbf{H}^{\mathsf{T}}\mathbf{C}_{\mathbf{z}}^{-1}\mathbf{H} + \mathbf{C}_{\overline{\mathbf{x}}}^{-1}$$

$$\hat{\mathbf{x}} = \mathbf{C}_{\hat{\mathbf{x}}}(\mathbf{H}^{\mathsf{T}}\mathbf{C}_{\mathbf{z}}^{-1}\mathbf{z} + \mathbf{C}_{\overline{\mathbf{x}}}^{-1}\overline{\mathbf{x}})$$

$$\rho = \frac{|\mathbf{C}_{\hat{\mathbf{x}}}|^{1/2}}{|\mathbf{C}_{\overline{\mathbf{x}}}|^{1/2}}\exp\left(-\frac{1}{2}d^2\right)$$

$$d^2 = (\mathbf{z}-\mathbf{H}\overline{\mathbf{x}})^{\mathsf{T}}(\mathbf{C_z} + \mathbf{H}\mathbf{C}_{\overline{\mathbf{x}}}\mathbf{H}^{\mathsf{T}})^{-1}(\mathbf{z}-\mathbf{H}\overline{\mathbf{x}}) \tag{A.3}$$

The Kalman filter Bayesian update is a normalized Gaussian product resulting in Gaussian PDF $\mathcal{N}(\mathbf{x},\hat{\mathbf{x}},\mathbf{C}_{\hat{\mathbf{x}}})$ in (A.3). This and the Markov projection in (37) and (38) are the complete Kalman filter solution [2,3]. Gaussian product factor ρ $(0<\rho<1)$ plays no role in the Kalman filter solution as contrasted to its important role in the more general problem here. Exponent d^2 is a form of the Mahalanobis distance.

Appendix B. Poisson PDF

Poisson PDF $\mathrm{p_{poi}}(m;\lambda)$ in (B.1) is discrete with non-negative integer random variable m and parameter λ.

$$\mathrm{p_{poi}}(m;\lambda) = \frac{\lambda^m \mathrm{e}^{-\lambda}}{m!}, \quad m = 0,1,2,\ldots$$

$$\mathrm{mean}(m) = \lambda, \quad \mathrm{var}(m) = \lambda \tag{B.1}$$

Appendix C. Gaussian mixture PDF

The Gaussian mixture PDF $\mathrm{p}(\mathbf{x})$ in (C.1) is a weighted average of N Gaussian PDF terms and is the canonical prior or posterior tracker PDF form at any scan with mean $\overline{\mathbf{x}}$ and covariance matrix $\mathbf{C_x}$. Gaussian PDF $\mathcal{N}()$ (see (A.1)) in each term n has mean $\overline{\mathbf{x}}_n$ and covariance matrix $\mathbf{C}_{\mathbf{x},n}$ and is a Gaussian "mode" with weight p_n. This form has been extensively used by others [33]:

$$\mathrm{p}(\mathbf{x}) = \sum_{n=1}^{N} p_n\mathcal{N}(\mathbf{x},\overline{\mathbf{x}}_n,\mathbf{C}_{\mathbf{x},n})$$

$$0 \le p_n \le 1, \quad \sum_{n=1}^{N} p_n = 1$$

$$\overline{\mathbf{x}} = \mathrm{E}[\mathbf{x}] = \sum_{n=1}^{N} p_n\overline{\mathbf{x}}_n$$

$$\mathbf{R_x} = \mathrm{E}[\mathbf{x}\mathbf{x}^{\mathsf{T}}] = \sum_{n=1}^{N} p_n(\overline{\mathbf{x}}_n\overline{\mathbf{x}}_n^{\mathsf{T}} + \mathbf{C}_{\mathbf{x},n})$$

$$\mathbf{C_x} = \mathrm{E}[(\mathbf{x}-\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})^{\mathsf{T}}] = \mathbf{R_x} - \overline{\mathbf{x}}\overline{\mathbf{x}}^{\mathsf{T}} \tag{C.1}$$

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